

ADELIC DIVISORS ON ARITHMETIC VARIETIES

ATSUSHI MORIWAKI

ABSTRACT. In this article, we generalize several fundamental results for arithmetic divisors, such as the continuity of the volume function, the generalized Hodge index theorem, Fujita's approximation theorem for arithmetic divisors and Zariski decompositions for arithmetic divisors on arithmetic surfaces, to the case of the adelic arithmetic divisors.

CONTENTS

| | |
|--|----|
| Introduction | 1 |
| 1. Preliminaries | 7 |
| 2. Adelic \mathbb{R} -Cartier divisors over a discrete valuation field | 16 |
| 3. Local and global density theorems | 29 |
| 4. Adelic arithmetic \mathbb{R} -Cartier divisors | 33 |
| 5. Continuity of the volume function | 46 |
| 6. Zariski decompositions of adelic arithmetic divisors on arithmetic surfaces | 56 |
| 7. Characterization of nef adelic arithmetic divisors on arithmetic surfaces | 67 |
| Appendix A. Characterization of relatively nef Cartier divisors | 75 |
| References | 83 |
| Index | 85 |

INTRODUCTION

The theory of birational Arakelov geometry has advanced tremendously over the last decade, such as the continuity of the volume function, the generalized Hodge index theorem, Fujita's approximation theorem for arithmetic divisors, Zariski decompositions for arithmetic divisors on arithmetic surfaces and so on. Besides them, non-Archimedean Arakelov geometry is also well-developed using Berkovich analytic spaces. In this article, we would like to generalize the fundamental results for arithmetic divisors to the case of adelic arithmetic divisors.

0.1. Birational Arakelov geometry. Let \mathcal{X} be a $(d+1)$ -dimensional, generically smooth, projective, normal arithmetic variety, that is, \mathcal{X} is a projective and flat normal integral scheme over \mathbb{Z} such that \mathcal{X} is smooth over \mathbb{Q} and the Krull dimension of \mathcal{X} is $d+1$. A pair $\mathcal{D} = (\mathcal{D}, g_\infty)$ is called an *arithmetic \mathbb{R} -Cartier divisor of C^0 -type on \mathcal{X}* if the following conditions are satisfied:

- (i) The first \mathcal{D} is an \mathbb{R} -Cartier divisor on \mathcal{X} , that is, $\mathcal{D} = a_1 \mathcal{D}_1 + \cdots + a_r \mathcal{D}_r$ for some Cartier divisors $\mathcal{D}_1, \dots, \mathcal{D}_r$ on \mathcal{X} and $a_1, \dots, a_r \in \mathbb{R}$.

- (ii) The second g_∞ is a real valued continuous function on $(\mathcal{X} \setminus \text{Supp}(\mathcal{D}))(\mathbb{C})$ such that, for each $x \in \mathcal{X}(\mathbb{C})$, $g_\infty + \sum_{i=1}^r a_i \log |f_i|^2$ extends to a continuous function around x , where f_1, \dots, f_r are local equations of $\mathcal{D}_1, \dots, \mathcal{D}_r$ at x , respectively. In addition, g_∞ is invariant under the complex conjugation map.

Let $\text{Rat}(\mathcal{X})$ be the rational function field of \mathcal{X} . We define $H^0(\mathcal{X}, \mathcal{D})$ to be

$$H^0(\mathcal{X}, \mathcal{D}) := \{\phi \in \text{Rat}(\mathcal{X})^\times \mid \mathcal{D} + (\phi) \geq 0\} \cup \{0\}.$$

Note that $H^0(\mathcal{X}, \mathcal{D})$ is a finitely generated \mathbb{Z} -module. For $\phi \in H^0(\mathcal{X}, \mathcal{D})$, we can see that $|\phi| \exp(-g_\infty/2)$ extends to a continuous function ϑ on $\mathcal{X}(\mathbb{C})$, so that

$$\sup \{\vartheta(x) \mid x \in \mathcal{X}(\mathbb{C})\}$$

is denoted by $\|\phi\|_{g_\infty}$. The volume $\widehat{\text{vol}}(\overline{\mathcal{D}})$ of $\overline{\mathcal{D}}$, by definition, is given by

$$\widehat{\text{vol}}(\overline{\mathcal{D}}) := \limsup_{n \rightarrow \infty} \frac{\log \# \{\phi \in H^0(\mathcal{X}, n\mathcal{D}) \mid \|\phi\|_{g_\infty} \leq 1\}}{n^{d+1}/(d+1)!}.$$

It is known that the volume function $\widehat{\text{vol}}$ has the following fundamental properties (for details, see [26]):

- (1) (Finiteness) $\widehat{\text{vol}}(\overline{\mathcal{D}}) < \infty$ ([22], [23]).
- (2) (Limit theorem) $\widehat{\text{vol}}(\overline{\mathcal{D}}) = \lim_{n \rightarrow \infty} \frac{\log \# \{\phi \in H^0(\mathcal{X}, n\mathcal{D}) \mid \|\phi\|_{g_\infty} \leq 1\}}{n^{d+1}/(d+1)!}$ ([7], [23]).
- (3) (Positive homogeneity) $\widehat{\text{vol}}(a\overline{\mathcal{D}}) = a^{d+1} \widehat{\text{vol}}(\overline{\mathcal{D}})$ for $a \in \mathbb{R}_{\geq 0}$ ([22], [23]).
- (4) (Continuity) The volume function $\widehat{\text{vol}}$ is continuous in the following sense: Let $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_r, \overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_s$ be arithmetic \mathbb{R} -Cartier divisors of C^0 -type on \mathcal{X} . For a compact subset B in \mathbb{R}^r and a positive number ϵ , there are positive numbers δ and δ' such that

$$\left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^s \delta_j \overline{\mathcal{A}}_j + (0, \phi) \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) \right| \leq \epsilon$$

for all $a_1, \dots, a_r, \delta_1, \dots, \delta_s \in \mathbb{R}$ and $\phi \in C^0(X)$ with $(a_1, \dots, a_r) \in B$, $|\delta_1| + \dots + |\delta_s| \leq \delta$ and $\|\phi\|_{\text{sup}} \leq \delta'$ ([22], [23]).

Here we would like to introduce several kinds of the positivity of an arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on \mathcal{X} .

- Big: $\widehat{\text{vol}}(\overline{\mathcal{D}}) > 0$.
- Relatively nef: the first Chern current $c_1(\overline{\mathcal{D}})$ is positive and \mathcal{D} is relatively nef with respect to $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$, that is, $\deg(\mathcal{D}|_C) \geq 0$ for all vertical 1-dimensional closed integral subschemes C of \mathcal{X} .
- Nef: $\overline{\mathcal{D}}$ is relatively nef and $\widehat{\deg}(\overline{\mathcal{D}}|_C) \geq 0$ for all horizontal 1-dimensional closed integral subschemes C of \mathcal{X} .

In addition, $\overline{\mathcal{D}}$ is said to be *integrable* if $\overline{\mathcal{D}} = \overline{\mathcal{D}}' - \overline{\mathcal{D}}''$ for some relatively nef arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}'$ and $\overline{\mathcal{D}}''$ of C^0 -type. For integrable arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}$ of C^0 -type, the arithmetic intersection number $\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1})$ is well-defined (cf. [26, Subsection 6.4], [27, Subsection 2.1]). The following fundamental results were obtained by several authors such as Faltings, Gillet-Soulé, S. Zhang, Moriwaki, H. Chen, X. Yuan and so on:

- (5) (Generalized Hodge index theorem) If $\overline{\mathcal{D}}$ is relatively nef, then

$$\widehat{\deg}(\overline{\mathcal{D}}^{d+1}) \leq \widehat{\text{vol}}(\overline{\mathcal{D}}).$$

Moreover, if $\overline{\mathcal{D}}$ is nef, then $\widehat{\deg}(\overline{\mathcal{D}}^{d+1}) = \widehat{\text{vol}}(\overline{\mathcal{D}})$ ([9], [10], [33], [22], [26]).

- (6) (Fujita's approximation theorem for arithmetic divisors) If $\overline{\mathcal{D}}$ is big, then, for any positive number ϵ , there are a birational morphism $\mu : \mathcal{Y} \rightarrow \mathcal{X}$ of generically smooth, normal and projective arithmetic varieties and a nef arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{Q}}$ of C^0 -type on \mathcal{Y} such that

$$\overline{\mathcal{Q}} \leq \mu^*(\overline{\mathcal{D}}) \quad \text{and} \quad \widehat{\text{vol}}(\overline{\mathcal{D}}) - \epsilon \leq \widehat{\text{vol}}(\overline{\mathcal{Q}}) \leq \widehat{\text{vol}}(\overline{\mathcal{D}})$$

([7], [31], [24]).

- (7) (Zariski decompositions for arithmetic divisors on arithmetic surfaces) We assume that $d = 1$ and \mathcal{X} is regular. Let $\Upsilon(\overline{\mathcal{D}})$ be the set of all nef arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{L}}$ of C^0 -type on \mathcal{X} with $\overline{\mathcal{L}} \leq \overline{\mathcal{D}}$. If $\Upsilon(\overline{\mathcal{D}}) \neq \emptyset$, then there is the greatest element $\overline{\mathcal{Q}}$ of $\Upsilon(\overline{\mathcal{D}})$, that is, $\overline{\mathcal{Q}} \in \Upsilon(\overline{\mathcal{D}})$ and $\overline{\mathcal{L}} \leq \overline{\mathcal{Q}}$ for all $\overline{\mathcal{L}} \in \Upsilon(\overline{\mathcal{D}})$ ([26], [28]).

The purpose of this article is to generalize the above results to adelic arithmetic divisors.

0.2. Green functions on analytic spaces over a complete discrete valuation field. Let k be a field and v a non-trivial complete discrete valuation of k . Let X be a projective and geometrically integral variety over k . Let X^{an} be the analytification of X in the sense of Berkovich [2]. Note that X^{an} is a compact Hausdorff space. Let $\text{Rat}(X)$ be the rational function field of X . Let D be an \mathbb{R} -Cartier divisor on X , that is, $D = a_1 D_1 + \cdots + a_r D_r$ for some Cartier divisors D_1, \dots, D_r on X and $a_1, \dots, a_r \in \mathbb{R}$. Let $X = \bigcup_{i=1}^N U_i$ be an affine open covering of X such that each D_j is given by $f_{ji} \in \text{Rat}(X)^\times$ on U_i for $j = 1, \dots, r$. We say a continuous function

$$g : X^{\text{an}} \setminus \bigcup_{j=1}^r \text{Supp}(D_j)^{\text{an}} \rightarrow \mathbb{R}$$

is a *D-Green function of C^0 -type on X^{an}* if $g + \sum_{j=1}^r a_j \log |f_{ji}|^2$ extends to a continuous function on U_i^{an} for each $i = 1, \dots, N$.

Let \mathcal{X} be a model of X over $\text{Spec}(k^\circ)$, that is, \mathcal{X} is a projective and flat integral scheme over $\text{Spec}(k^\circ)$ such that the generic fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ is X , where

$$k^\circ := \{f \in k \mid v(f) \leq 1\}.$$

We assume that there are Cartier divisors $\mathcal{D}_1, \dots, \mathcal{D}_r$ on \mathcal{X} such that $\mathcal{D}_j \cap X = D_j$ for $j = 1, \dots, r$. We set $\mathcal{D} := a_1 \mathcal{D}_1 + \cdots + a_r \mathcal{D}_r$. The pair $(\mathcal{X}, \mathcal{D})$ is called a *model of (X, D)* . For $x \in X^{\text{an}} \setminus \bigcup_{j=1}^r \text{Supp}(D_j)^{\text{an}}$, let f_1, \dots, f_r be local equations of $\mathcal{D}_1, \dots, \mathcal{D}_r$ at $\xi = r_{\mathcal{X}}(x)$, respectively, where $r_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_\circ$ is the reduction map and \mathcal{X}_\circ is the central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$. We define $g_{(\mathcal{X}, \mathcal{D})}(x)$ to be

$$g_{(\mathcal{X}, \mathcal{D})}(x) := - \sum_{j=1}^r a_j \log |f_j(x)|^2.$$

It is easy to see that $g_{(\mathcal{X}, \mathcal{D})}$ is a *D-Green function of C^0 -type on X^{an}* . We call it the *Green function induced by the model $(\mathcal{X}, \mathcal{D})$* .

We say a *D-Green function g is of $(C^0 \cap \text{PSH})$ -type* if D is nef and there is a sequence $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^\infty$ of models of (X, D) with the following properties:

- (i) For each $n \geq 1$, \mathcal{D}_n is relatively nef with respect to $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$.
- (ii) If we set $\phi_n = g(\mathcal{X}_n, \mathcal{D}_n) - g$, then $\lim_{n \rightarrow \infty} \|\phi_n\|_{\text{sup}} = 0$.

0.3. Adelic arithmetic divisors. Let K be a number field and O_K the ring of integers in K . We denote the set of all maximal ideals of O_K by M_K . For $P \in M_K$, the valuation v_P of K at P is given by

$$v_P(f) = \#(O_K/P)^{-\text{ord}_P(f)}.$$

Let K_P be the completion of K with respect to v_P . Let X be a d -dimensional, projective, smooth and geometrically integral variety over K and let $X_P := X \times_{\text{Spec}(K)} \text{Spec}(K_P)$, which is also a projective, smooth and geometrically integral variety over K_P . Let $X(\mathbb{C})$ be the set of all \mathbb{C} -valued points of X , that is,

$$X(\mathbb{C}) := \{x : \text{Spec}(\mathbb{C}) \rightarrow X \mid x \text{ is a morphism as schemes}\}.$$

Let $F_\infty : X(\mathbb{C}) \rightarrow X(\mathbb{C})$ be the complex conjugation map, that is, for $x \in X(\mathbb{C})$, $F_\infty(x)$ is given by the composition of morphisms $\text{Spec}(\mathbb{C}) \xrightarrow{-a} \text{Spec}(\mathbb{C})$ and $\text{Spec}(\mathbb{C}) \xrightarrow{x} X$, where $\text{Spec}(\mathbb{C}) \xrightarrow{-a} \text{Spec}(\mathbb{C})$ is the morphism induced by the complex conjugation. The space of F_∞ -invariant real valued continuous functions on $X(\mathbb{C})$ is denoted by $C_{F_\infty}^0(X(\mathbb{C}))$, that is,

$$C_{F_\infty}^0(X(\mathbb{C})) := \{\varphi \in C^0(X(\mathbb{C})) \mid \varphi \circ F_\infty = \varphi\}.$$

A pair $\overline{D} = (D, g)$ of an \mathbb{R} -Cartier divisor D on X and a collection of Green functions

$$g = \{g_P\}_{P \in M_K} \cup \{g_\infty\}$$

is called an *adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X* if the following conditions are satisfied:

- (1) For each $P \in M_K$, g_P is a D -Green function of C^0 -type on X_P^{an} . In addition, there are a non-empty open set U of $\text{Spec}(O_K)$, a normal model \mathcal{X}_U of X over U and an \mathbb{R} -Cartier divisor \mathcal{D}_U on \mathcal{X}_U such that $\mathcal{D}_U \cap X = D$ and g_P is a D -Green function induced by the model $(\mathcal{X}_U, \mathcal{D}_U)$ for all $P \in U \cap M_K$.
- (2) The function g_∞ is an F_∞ -invariant D -Green function of C^0 -type on $X(\mathbb{C})$.

For simplicity, a collection of Green functions $g = \{g_P\}_{P \in M_K} \cup \{g_\infty\}$ is often expressed by the following symbol:

$$g = \sum_{P \in M_K} g_P[P] + g_\infty[\infty].$$

We denote the space of all adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X by $\widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$.

Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . We define $H^0(X, D)$ to be

$$H^0(X, D) := \{\phi \in \text{Rat}(X)^\times \mid D + (\phi) \geq 0\} \cup \{0\}.$$

For $\phi \in H^0(X, D)$ and $\wp \in M_K \cup \{\infty\}$, $|\phi| \exp(-g_\wp/2)$ extends to a continuous function, so that its supremum is denoted by $\|\phi\|_{g_\wp}$. The set $\hat{H}^0(X, \overline{D})$ of small sections of \overline{D} and the volume $\widehat{\text{vol}}(\overline{D})$ of \overline{D} are defined by

$$\begin{cases} \hat{H}^0(X, \overline{D}) := \{\phi \in H(X, D) \mid \|\phi\|_{g_\wp} \leq 1 \text{ for all } \wp \in M_K \cup \{\infty\}\}, \\ \widehat{\text{vol}}(\overline{D}) := \limsup_{n \rightarrow \infty} \frac{\log \# \hat{H}^0(X, n\overline{D})}{n^{d+1}/(d+1)!}, \end{cases}$$

respectively. Similarly as given for arithmetic divisors, we can also introduce several kinds of the positivity of \overline{D} as follows:

- Big: $\widehat{\text{vol}}(\overline{D}) > 0$.
- Relatively nef: g_P is of $(C^0 \cap \text{PSH})$ -type for all $P \in M_K$ and the first Chern current $c_1(D, g_\infty)$ is positive.
- Nef: \overline{D} is relatively nef and the height function arising from \overline{D} is non-negative.

Further, \overline{D} is said to be *integrable* if $\overline{D} = \overline{D}' - \overline{D}''$ for some relatively nef adelic arithmetic \mathbb{R} -Cartier divisors \overline{D}' and \overline{D}'' of C^0 -type on X . For integrable adelic arithmetic \mathbb{R} -Cartier divisors $\overline{D}_1, \dots, \overline{D}_{d+1}$ of C^0 -type on X , the arithmetic intersection number

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1})$$

can be defined (cf. Subsection 4.5).

0.4. Main results. Let X be a d -dimensional, projective, smooth and geometrically integral variety over a number field K . The following theorems are the main results of this article. Theorem 0.4.1, Theorem 0.4.2, Theorem 0.4.3 and Theorem 0.4.4 are generalizations of (4), (5), (6) and (7), respectively. The properties (1), (2) and (3) also hold for adelic arithmetic divisors (cf. Theorem 5.1.1). Several similar results on arithmetic toric varieties are known. For details, see [3] and [4]. The adelic version of Fujita's approximation theorem has been already established by Boucksom and Chen [5]. In this article, we give another proof of it and generalize it to \mathbb{R} -divisors. Further, Theorem 0.4.5 is a generalization of the result proved in [28].

Theorem 0.4.1 (Continuity of the volume function for adelic arithmetic divisors). *The volume function $\widehat{\text{vol}} : \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}} \rightarrow \mathbb{R}$ is continuous in the following sense: Let $\overline{D}_1, \dots, \overline{D}_r, \overline{A}_1, \dots, \overline{A}_{r'}$ be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Let $\{P_1, \dots, P_s\}$ be a finite subset of M_K . For a compact subset B in \mathbb{R}^r and a positive number ϵ , there are positive numbers δ and δ' such that*

$$\left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{P_l}[P_l] + \varphi_\infty[\infty] \right) \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \leq \epsilon$$

holds for all $a_1, \dots, a_r, \delta_1, \dots, \delta_{r'} \in \mathbb{R}$, $\varphi_{P_1} \in C^0(X_{P_1}^{\text{an}}), \dots, \varphi_{P_s} \in C^0(X_{P_s}^{\text{an}})$ and $\varphi_\infty \in C_{F_\infty}^0(X(\mathbb{C}))$ with $(a_1, \dots, a_r) \in B$, $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\sum_{l=1}^s \|\varphi_{P_l}\|_{\text{sup}} + \|\varphi_\infty\|_{\text{sup}} \leq \delta'$.

Theorem 0.4.2 (Generalized Hodge index theorem for adelic arithmetic divisors). *Let \overline{D} be a relatively nef adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Then*

$$\widehat{\deg}(\overline{D}^{d+1}) \leq \widehat{\text{vol}}(\overline{D}).$$

Moreover, if \overline{D} is nef, then $\widehat{\deg}(\overline{D}^{d+1}) = \widehat{\text{vol}}(\overline{D})$.

Theorem 0.4.3 (Fujita's approximation theorem for adelic arithmetic divisors). *Let \overline{D} be a big adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Then, for any positive number ϵ , there are a birational morphism $\mu : Y \rightarrow X$ of smooth, projective and geometrically integral varieties over K and a nef adelic arithmetic \mathbb{R} -Cartier divisor \overline{Q} of C^0 -type on Y such that $\overline{Q} \leq \mu^*(\overline{D})$ and $\widehat{\text{vol}}(\overline{D}) - \epsilon \leq \widehat{\text{vol}}(\overline{Q}) \leq \widehat{\text{vol}}(\overline{D})$.*

Theorem 0.4.4 (Zariski decompositions for adelic arithmetic divisors on curves). *We assume $d = 1$. Let \overline{D} be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Let $\Upsilon(\overline{D})$ be the set of all nef adelic arithmetic \mathbb{R} -Cartier divisors \overline{L} of C^0 -type on X with $\overline{L} \leq \overline{D}$. If $\Upsilon(\overline{D}) \neq \emptyset$, then there is the greatest element \overline{Q} of $\Upsilon(\overline{D})$, that is, $\overline{Q} \in \Upsilon(\overline{D})$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon(\overline{D})$. Moreover, the natural map $\hat{H}^0(X, a\overline{Q}) \rightarrow \hat{H}^0(X, a\overline{D})$ is bijective for $a \in \mathbb{R}_{>0}$. In particular, $\widehat{\text{vol}}(\overline{Q}) = \widehat{\text{vol}}(\overline{D})$.*

Theorem 0.4.5 (Numerical characterization of nef adelic arithmetic divisors on curves). *We assume $d = 1$. Let \overline{D} be an integrable adelic arithmetic \mathbb{R} -Cartier divisor on X . Then \overline{D} is nef if and only if \overline{D} is pseudo-effective and $\widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}(\overline{D})$.*

0.5. Conventions and terminology.

0.5.1. For a topological space M , the set of all real valued continuous functions on M is denote by $C^0(M)$. Note that $C^0(M)$ forms an \mathbb{R} -algebra.

0.5.2. Let k be a field and v a non-Archimedean valuation of k . We define k° and k^∞ to be

$$k^\circ := \{x \in k \mid v(x) \leq 1\} \quad \text{and} \quad k^\infty := \{x \in k \mid v(x) < 1\}.$$

Note that k° is a valuation ring and k^∞ is its maximal ideal. If v is discrete and complete, then k° is excellent.

0.5.3. Let M be a finitely generated \mathbb{Z} -module and let $\|\cdot\|$ be a norm of $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. We define $\hat{h}^0(M, \|\cdot\|)$ and $\hat{\chi}(M, \|\cdot\|)$ to be

$$\begin{cases} \hat{h}^0(M, \|\cdot\|) := \log \# \{x \in M \mid \|x\| \leq 1\}, \\ \hat{\chi}(M, \|\cdot\|) := \log \left(\frac{\text{vol}(B(M, \|\cdot\|))}{\text{vol}(M_{\mathbb{R}}/(M/M_{\text{tor}}))} \right) + \log \#(M_{\text{tor}}), \end{cases}$$

where $B(M, \|\cdot\|)$ is the unit ball with respect to $\|\cdot\|$ (i.e. $B(M, \|\cdot\|) := \{x \in M_{\mathbb{R}} \mid \|x\| \leq 1\}$), M_{tor} is the torsion subgroup of M and $\text{vol}(M_{\mathbb{R}}/(M/M_{\text{tor}}))$ is the volume of the fundamental domain of $M_{\mathbb{R}}/(M/M_{\text{tor}})$.

0.5.4. Let S be a noetherian integral scheme. An integral scheme X over S is called a *variety over S* if X is flat, separated and of finite type over S . If S is given by $\text{Spec}(O_K)$ (i.e. K is a number field and O_K is the ring of integers in K), then a variety over S is often called an *arithmetic variety*.

0.5.5. Let S be a noetherian integral scheme and k the rational function field of S . Let X be a projective variety over k . A projective variety \mathcal{X} over S is called a *model of X over S* if the generic fiber of $\mathcal{X} \rightarrow S$ is X . Moreover, if \mathcal{X} are normal (resp. regular), then \mathcal{X} is called a *normal model of X over S* (resp. *regular model of X over S*). Note that if \mathcal{X} is normal (resp. regular), then X is also normal (resp. regular). We assume that S is an excellent Dedekind scheme, $\dim X = 1$ and X is smooth over k . By [18], for any model \mathcal{X} of X over S , there is a regular model \mathcal{X}' of X over S together with a birational morphism $\mathcal{X}' \rightarrow \mathcal{X}$.

0.5.6. Let $f : \mathcal{X} \rightarrow S$ be a proper morphism of noetherian schemes. Let C be a curve on \mathcal{X} , that is, a 1-dimensional reduced and irreducible closed subscheme on \mathcal{X} . The curve C is said to be *vertical* with respect to $f : \mathcal{X} \rightarrow S$ if $f(C)$ is a closed point of S . For $\mathcal{L} \in \text{Pic}(\mathcal{X}) \otimes \mathbb{R}$, we say \mathcal{L} is *relatively nef* with respect to $f : \mathcal{X} \rightarrow S$ if $\deg(\mathcal{L}|_C) \geq 0$ for all vertical curves C on \mathcal{X} . Let \mathcal{D} be an \mathbb{R} -Cartier divisor on \mathcal{X} , that is, $\mathcal{D} = a_1 \mathcal{D}_1 + \dots + a_r \mathcal{D}_r$ for some Cartier divisors $\mathcal{D}_1, \dots, \mathcal{D}_r$ on \mathcal{X} and $a_1, \dots, a_r \in \mathbb{R}$. The \mathbb{R} -Cartier divisor \mathcal{D} is said to be *relatively nef* with respect to $f : \mathcal{X} \rightarrow S$ if $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_1)^{\otimes a_1} \otimes \dots \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D}_r)^{\otimes a_r} \in \text{Pic}(\mathcal{X}) \otimes \mathbb{R}$ is relatively nef with respect to $f : \mathcal{X} \rightarrow S$.

0.5.7. Let (A, m) be a 1-dimensional noetherian local domain. For $x \in A \setminus \{0\}$, we define $\text{ord}_A(x)$ to be $\text{ord}_A(x) := \text{length}_A(A/xA)$. It is easy to see that

$$\text{ord}_A(xy) = \text{ord}_A(x) + \text{ord}_A(y)$$

for $x, y \in A \setminus \{0\}$, so that it extends to F^\times as a homomorphism, where F is the quotient field of A . Further, if we set $F_{\mathbb{R}}^\times := F^\times \otimes_{\mathbb{Z}} \mathbb{R}$, then ord_A also extends to $F_{\mathbb{R}}^\times$. Let X be a noetherian integral scheme and γ a point of X such that $\dim \mathcal{O}_{X, \gamma} = 1$. Then $\text{ord}_{\mathcal{O}_{X, \gamma}}$ is often denoted by ord_γ or ord_Γ , where Γ is the closure of $\{\gamma\}$.

0.5.8. Let X be a regular scheme and let $\text{Div}(X)$ be the group of Cartier divisors on X . We set $\text{Div}(X)_{\mathbb{R}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an \mathbb{R} -Cartier divisor. As X is regular, an \mathbb{R} -Cartier divisor D has a unique expression

$$D = \sum_{\Gamma} a_{\Gamma} \Gamma,$$

where $a_{\Gamma} \in \mathbb{R}$ and Γ runs over all prime divisors on X . For $D_1, \dots, D_r \in \text{Div}(X)_{\mathbb{R}}$, we set

$$D_1 = \sum_{\Gamma} a_{1, \Gamma} \Gamma, \dots, D_r = \sum_{\Gamma} a_{r, \Gamma} \Gamma.$$

We define $\max\{D_1, \dots, D_r\}$ and $\min\{D_1, \dots, D_r\}$ to be

$$\begin{cases} \max\{D_1, \dots, D_r\} := \sum_{\Gamma} \max\{a_{1, \Gamma}, \dots, a_{r, \Gamma}\} \Gamma, \\ \min\{D_1, \dots, D_r\} := \sum_{\Gamma} \min\{a_{1, \Gamma}, \dots, a_{r, \Gamma}\} \Gamma. \end{cases}$$

1. PRELIMINARIES

The goal of this section is to prepare several kinds of materials for the later sections. In Subsection 1.1, we consider the support of an \mathbb{R} -Cartier divisor. In Subsection 1.2, we quickly review the analytification of an algebraic scheme in the sense of Berkovich [2]. Subsection 1.3 is devoted to the proof of several lemmas.

1.1. \mathbb{R} -Cartier divisors on a noetherian integral scheme. Let A be a noetherian integral domain and F the quotient field of A . Let \mathbb{K} be either \mathbb{Z} or \mathbb{Q} or \mathbb{R} . We set

$$F_{\mathbb{K}}^\times := (F^\times, \times) \otimes_{\mathbb{Z}} \mathbb{K} \quad \text{and} \quad (A_p^\times)_{\mathbb{K}} := (A_p^\times, \times) \otimes_{\mathbb{Z}} \mathbb{K}$$

for $p \in \text{Spec}(A)$. As \mathbb{K} is flat over \mathbb{Z} , we have $(A_p^\times)_{\mathbb{K}} \subseteq F_{\mathbb{K}}^\times$. For $f \in F_{\mathbb{K}}^\times$, we define $V_{\mathbb{K}}(f)$ to be

$$V_{\mathbb{K}}(f) := \left\{ p \in \text{Spec}(A) \mid f \notin (A_p^\times)_{\mathbb{K}} \right\}.$$

Let us begin with the following proposition:

- Proposition 1.1.1.** (1) $V_{\mathbb{R}}(f) = V_{\mathbb{Q}}(f)$ for $f \in F_{\mathbb{Q}}^{\times}$.
 (2) Let $f \in F^{\times}$. Then $V_{\mathbb{Q}}(f) = \bigcap_{n=1}^{\infty} V_{\mathbb{Z}}(f^n)$. Moreover, if A is normal, then $V_{\mathbb{Q}}(f) = V_{\mathbb{Z}}(f)$.
 (3) For $f \in F_{\mathbb{K}}^{\times}$, the set $V_{\mathbb{K}}(f)$ is closed in $\text{Spec}(A)$.

Proof. (1) By Lemma 1.3.1 in Subsection 1.3, $(A_p^{\times})_{\mathbb{Q}} = F_{\mathbb{Q}}^{\times} \cap (A_p^{\times})_{\mathbb{R}}$, and hence (1) follows.

(2) As $F_{\mathbb{Q}}^{\times}/(A_p^{\times})_{\mathbb{Q}} = (F^{\times}/A_p^{\times}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $f = 1$ in $F_{\mathbb{Q}}^{\times}/(A_p^{\times})_{\mathbb{Q}}$ if and only if $f^n = 1$ in F^{\times}/A_p^{\times} for some $n \in \mathbb{Z}_{>0}$. Thus the first assertion follows. The second assertion is obvious because $V_{\mathbb{Z}}(f) = V_{\mathbb{Z}}(f^n)$ if A is normal.

(3) First we prove that $V_{\mathbb{Z}}(f)$ is closed for $f \in F^{\times}$. We set

$$I = \{a \in A \mid af \in A\} \quad \text{and} \quad J = If.$$

Clearly I and J are ideals of A . Note that $I_p = \{a \in A_p \mid af \in A_p\}$ by [1, Corollary 3.15]). Thus,

$$f \in A_p^{\times} \iff I_p = A_p \text{ and } J_p = A_p,$$

so that $V_{\mathbb{Z}}(f) = \text{Supp}(\text{Spec}(A/I)) \cup \text{Supp}(\text{Spec}(A/J))$, which is closed.

Next let us see that $V(f)_{\mathbb{Q}}$ is closed for $f \in F_{\mathbb{Q}}^{\times}$. Clearly we may assume that $f \in F^{\times}$ because, for $n \in \mathbb{Z}_{>0}$, $f \in (A_p^{\times})_{\mathbb{Q}}$ if and only if $f^n \in (A_p^{\times})_{\mathbb{Q}}$. Thus, by (2), $V_{\mathbb{Q}}(f)$ is closed.

Finally we consider the case $\mathbb{K} = \mathbb{R}$. We can find $f_1, \dots, f_r \in F^{\times}$ and $a_1, \dots, a_r \in \mathbb{R}$ such that $f = f_1^{a_1} \cdots f_r^{a_r}$ and a_1, \dots, a_r are linearly independent over \mathbb{Q} . Then, by Lemma 1.3.1,

$$f \in (A_p^{\times})_{\mathbb{R}} \iff f_1, \dots, f_r \in (A_p^{\times})_{\mathbb{Q}},$$

and hence $V_{\mathbb{R}}(f) = \bigcup_{i=1}^r V_{\mathbb{Q}}(f_i)$, which is closed by the previous observation. \square

Definition 1.1.2. Let X be a noetherian integral scheme and let $\text{Rat}(X)$ be the rational function field of X . Let $\text{Div}(X)$ be the group of Cartier divisors on X , that is,

$$\text{Div}(X) := H^0(X, \text{Rat}(X)^{\times} / \mathcal{O}_X^{\times}).$$

Let \mathbb{K} be either \mathbb{Z} or \mathbb{Q} or \mathbb{R} . We set

$$\text{Div}(X)_{\mathbb{K}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K} \quad \text{and} \quad \text{Rat}(X)_{\mathbb{K}}^{\times} := \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{K}.$$

An element of $\text{Div}(X)_{\mathbb{K}}$ (reps. $\text{Rat}(X)_{\mathbb{K}}^{\times}$) is called a \mathbb{K} -Cartier divisor on X (reps. \mathbb{K} -rational function on X). A \mathbb{K} -rational function $f \in \text{Rat}(X)_{\mathbb{K}}^{\times}$ naturally gives rise to a \mathbb{K} -Cartier divisor, which is called the \mathbb{K} -principal divisor of f and is denoted by $(f)_{\mathbb{K}}$. Occasionally, $(f)_{\mathbb{K}}$ is denoted by (f) for simplicity. For $D \in \text{Div}(X)_{\mathbb{K}}$ (i.e., $D = a_1 D_1 + \cdots + a_r D_r$ for some $D_1, \dots, D_r \in \text{Div}(X)$ and $a_1, \dots, a_r \in \mathbb{K}$), there is an affine open covering $X = \bigcup_{i=1}^N \text{Spec}(A_i)$ of X such that D is given by some $f_i \in \text{Rat}(X)_{\mathbb{K}}^{\times}$ and $f_i/f_j \in (\mathcal{O}_{X,p}^{\times})_{\mathbb{K}} := \mathcal{O}_{X,p}^{\times} \otimes_{\mathbb{Z}} \mathbb{K}$ for all $p \in U_i \cap U_j$, so that $V_{\mathbb{K}}(f_i) = V_{\mathbb{K}}(f_j)$ on $U_i \cap U_j$, where $U_i = \text{Spec}(A_i)$ for $i = 1, \dots, N$. Therefore, we have a closed set Z on X such that $Z|_{U_i} = V_{\mathbb{K}}(f_i)$ for all $i = 1, \dots, N$. It is called the \mathbb{K} -support of D and is denoted by $\text{Supp}_{\mathbb{K}}(D)$. By Proposition 1.1.1, $\text{Supp}_{\mathbb{R}}(D) = \text{Supp}_{\mathbb{Q}}(D)$ for $D \in \text{Div}(X)_{\mathbb{Q}}$ and $\text{Supp}_{\mathbb{Q}}(D) = \bigcap_{n=1}^{\infty} \text{Supp}_{\mathbb{Z}}(nD)$ for $D \in \text{Div}(X)$.

From now on, we assume that X is normal. Then $\text{Supp}_{\mathbb{Q}}(D) = \text{Supp}_{\mathbb{Z}}(D)$ for $D \in \text{Div}(X)$. For a \mathbb{K} -Cartier divisor D on X , the associated \mathbb{K} -Weil divisor D_W of D is defined by

$$D_W := \sum_{\Gamma : \text{prime divisor}} \text{ord}_{\Gamma}(f_{\Gamma}) \Gamma,$$

where f_Γ is a local equation of D at Γ . The *support of D as a Weil-divisor* is denoted by $\text{Supp}_W(D)$, that is,

$$\text{Supp}_W(D) := \bigcup_{\text{ord}_\Gamma(f_\Gamma) \neq 0} \Gamma.$$

Proposition 1.1.3. *We assume that X is normal. Let D be a \mathbb{K} -Cartier divisor on X . Then $\text{Supp}_W(D) \subseteq \text{Supp}_\mathbb{K}(D)$. Further, if X is regular, then $\text{Supp}_W(D) = \text{Supp}_\mathbb{K}(D)$.*

Proof. We use the same notation as in Definition 1.1.2. Let $p \in U_i \setminus \text{Supp}_\mathbb{K}(D)$. Then $f_i \in (\mathcal{O}_{X,p}^\times)_\mathbb{K}$. In particular, $\text{ord}_\Gamma(f_i) = 0$ for all prime divisors Γ with $p \in \Gamma$, and hence $p \notin \text{Supp}_W(D)$, as desired.

We assume that X is regular. Let $p \in U_i \cap \text{Supp}_\mathbb{K}(D)$. As $\mathcal{O}_{X,p}$ is a UFD, there are distinct prime elements $h_1, \dots, h_r \in \mathcal{O}_{X,p}$ modulo $\mathcal{O}_{X,p}^\times$, $u \in (\mathcal{O}_{X,p}^\times)_\mathbb{K}$ and $a_1, \dots, a_r \in \mathbb{R}$ such that $f_i = u h_1^{a_1} \cdots h_r^{a_r}$. If $a_1 = \cdots = a_r = 0$, then $f_i \in (\mathcal{O}_{X,p}^\times)_\mathbb{K}$, which contracts to $p \in U_i \cap \text{Supp}_\mathbb{K}(D)$, so that we may assume that $a_1, \dots, a_r \in \mathbb{R}_{\neq 0}$. Since h_1, \dots, h_r are distinct modulo $\mathcal{O}_{X,p}^\times$, $\Gamma_1 = \text{Spec}(\mathcal{O}_{X,p}/h_1), \dots, \Gamma_r = \text{Spec}(\mathcal{O}_{X,p}/h_r)$ give rise to distinct prime divisors. In addition, $\text{ord}_{\Gamma_j}(f_i) = a_j$ for $j = 1, \dots, r$. Therefore, $p \in \text{Supp}_W(D)$. \square

Finally let us consider Hartogs' lemma for \mathbb{R} -rational functions.

Lemma 1.1.4 (Hartogs' lemma for \mathbb{R} -rational functions). *Let A be a normal and noetherian domain and F the quotient field of A . For $x \in F_\mathbb{R}^\times$, if $\text{ord}_\Gamma(x) \geq 0$ for all prime divisors Γ of A , then there are $x_1, \dots, x_r \in A \setminus \{0\}$ and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ with $x = x_1^{a_1} \cdots x_r^{a_r}$.*

Proof. Let Σ be the set of all prime divisors of A . Clearly we can find $y_1, \dots, y_r \in F^\times$ and $c_1, \dots, c_r \in \mathbb{R}_{>0}$ such that $x = y_1^{c_1} \cdots y_r^{c_r}$ and c_1, \dots, c_r are linearly independent over \mathbb{Q} . We set $\Sigma' = \{\Gamma \in \Sigma \mid \text{ord}_\Gamma(x) > 0\}$. Let us see the following:

Claim 1.1.4.1. *Σ' is a finite set and $\text{ord}_\Gamma(y_i) = 0$ for all $\Gamma \in \Sigma \setminus \Sigma'$ and $i = 1, \dots, r$.*

Proof. If $\Gamma \notin \text{Supp}_\mathbb{Z}((x)_\mathbb{R})$, then $\text{ord}_\Gamma(x) = 0$, so that Σ' is a finite set. Moreover, if $\Gamma \in \Sigma \setminus \Sigma'$, then

$$0 = \text{ord}_\Gamma(x) = c_1 \text{ord}_\Gamma(y_1) + \cdots + c_r \text{ord}_\Gamma(y_r),$$

and hence $\text{ord}_\Gamma(y_1) = \cdots = \text{ord}_\Gamma(y_r) = 0$ by the linear independency of c_1, \dots, c_r over \mathbb{Q} . \square

By virtue of Lemma 1.3.2 in Subsection 1.3, there are $e_{ij} \in \mathbb{Q}_{>0}$ ($i, j = 1, \dots, r$) and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ such that, if we set $x_i = y_1^{e_{i1}} \cdots y_r^{e_{ir}}$ for $i = 1, \dots, r$, then $x = x_1^{a_1} \cdots x_r^{a_r}$ and $\text{ord}_\Gamma(x_i) > 0$ for all $\Gamma \in \Sigma'$ and $i = 1, \dots, r$. Replacing e_{ij} by $e e_{ij}$ and a_i by a_i/e for some $e \in \mathbb{Z}_{>0}$, we may assume that $e_{ij} \in \mathbb{Z}$ for all i, j . In particular, $x_i \in F^\times$ for $i = 1, \dots, r$. Note that, for $\Gamma \in \Sigma \setminus \Sigma'$,

$$\text{ord}_\Gamma(x_i) = e_{i1} \text{ord}_\Gamma(y_1) + \cdots + e_{ir} \text{ord}_\Gamma(y_r) = 0,$$

and hence $\text{ord}_\Gamma(x_i) \geq 0$ for all $\Gamma \in \Sigma$ and $i = 1, \dots, r$. Therefore, by algebraic Hartogs' lemma, $x_i \in A \setminus \{0\}$ for $i = 1, \dots, r$, as required. \square

1.2. Analytification of algebraic schemes over a complete valuation field. Throughout this subsection, k is a field and v is a complete valuation of k . Here we quickly review the analytification of algebraic schemes over k in the sense of Berkovich [2].

Let A be a k -algebra. We say a map $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$ is a *multiplicative semi-norm over k* if the following conditions are satisfied:

- (1) $|a + b| \leq |a| + |b|$ for all $a, b \in A$.
- (2) $|ab| = |a||b|$ for all $a, b \in A$.
- (3) $|a| = v(a)$ for all $a \in k$.

Let $x = |\cdot|_x$ be a multiplicative semi-norm over k . We set $p_x := \{a \in A \mid |a|_x = 0\}$, which is a prime ideal of A . We call p_x the *associated prime of x* . The residue field at p_x is denoted by $k(x)$. Clearly x descends to a valuation v_x of $k(x)$ such that $v_x(a) = v(a)$ for all $a \in k$. The field $k(x)$ and the valuation v_x are called the *residue field of x* and the *associated valuation of x* , respectively. Conversely, let v' be a valuation of the residue field $k(p)$ at $p \in \text{Spec}(A)$ such that $v'(a) = v(a)$ for all $a \in k$. If we set $|a| := v'(a \bmod p)$ for $a \in A$, then $|\cdot|$ yields a multiplicative semi-norm over k whose residue field and associated valuation are $k(p)$ and v' , respectively. In particular, this observation shows that if v is non-Archimedean, then $|\cdot|$ is also non-Archimedean, that is, $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in A$.

We denote the set of all multiplicative semi-norms over k by $\text{Spec}_k^{\text{an}}(A)$. For $x = |\cdot|_x \in \text{Spec}_k^{\text{an}}(A)$, $|a|_x$ is often denoted by $|a(x)|$. We equip the weakest topology to $\text{Spec}_k^{\text{an}}(A)$ such that the map $\text{Spec}_k^{\text{an}}(A) \rightarrow \mathbb{R}_{\geq 0}$ given by $x \mapsto |a(x)|$ is continuous for every $a \in A$, that is, the collection

$$\left\{ \{x \in \text{Spec}_k^{\text{an}}(A) \mid |a(x)| \in U\} \right\}_{a, U} \quad (\text{where } a \in A \text{ and } U \text{ is an open set in } \mathbb{R}_{\geq 0})$$

forms a subbasis of the topology. A map $\text{Spec}_k^{\text{an}}(A) \rightarrow \text{Spec}(A)$ given by $x \mapsto p_x$ is denoted by p . It is easy to see that $p : \text{Spec}_k^{\text{an}}(A) \rightarrow \text{Spec}(A)$ is continuous.

Let $f : A \rightarrow B$ be a homomorphism of k -algebras. We define a map

$$f^{\text{an}} : \text{Spec}_k^{\text{an}}(B) \rightarrow \text{Spec}_k^{\text{an}}(A)$$

to be $|a|_{f^{\text{an}}(y)} = |f(a)|_y$ for $y = |\cdot|_y \in \text{Spec}_k^{\text{an}}(B)$ and $a \in A$. We can easily check that $f^{\text{an}} : \text{Spec}_k^{\text{an}}(B) \rightarrow \text{Spec}_k^{\text{an}}(A)$ is continuous. Let s be a non-nilpotent element of A . Let $\iota : A \rightarrow A_s$ be the canonical homomorphism. Then we can see that ι^{an} yields a homeomorphism

$$(1.2.1) \quad \text{Spec}_k^{\text{an}}(A_s) \xrightarrow[\text{homeo}]{\approx} \{x \in \text{Spec}_k^{\text{an}}(A) \mid |s(x)| \neq 0\}.$$

Let X be an algebraic scheme over k , that is, a scheme separated and of finite type over k . If $X = \text{Spec}(A)$ is an affine scheme over k , then $X^{\text{an}} := \text{Spec}_k^{\text{an}}(A)$. In general, if $X = \bigcup_{i=1}^N U_i$ is an affine open covering of X , then X^{an} is defined by gluing together U_i^{an} as a topological space (cf. (1.2.1)). For each i , we can define $p : U_i^{\text{an}} \rightarrow U_i$, which can be extended to a continuous map $p : X^{\text{an}} \rightarrow X$. Let $f : X \rightarrow Y$ be a morphism of algebraic schemes over k . We can see that f induces a natural continuous map $f^{\text{an}} : X^{\text{an}} \rightarrow Y^{\text{an}}$.

From now on, we assume that v is non-Archimedean and X is proper over k . Let \mathcal{X} be a proper and flat scheme over $\text{Spec}(k^\circ)$ such that the generic fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ is X . Let \mathcal{X}_\circ be the central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$, that is, $\mathcal{X}_\circ = \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k^\circ/k^{\circ\circ})$ (for the definitions of k° and $k^{\circ\circ}$, see Conventions and terminology 0.5.2). Let

$$r_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_\circ$$

be the *reduction map* induced by $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$, which can be defined in the following way: For $x \in X^{\text{an}}$, let $k(x)$ be the residue field of x . Then, by using the valuation criterion

of properness, there is a morphism $t : \text{Spec}(k(x)^\circ) \rightarrow \mathcal{X}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \text{Spec}(k(x)) & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow t & \downarrow \\ \text{Spec}(k(x)^\circ) & \longrightarrow & \text{Spec}(k^\circ) \end{array}$$

Then $r_{\mathcal{X}}(x)$ is given by $t(k(x)^\circ)$. The morphism $t : \text{Spec}(k(x)^\circ) \rightarrow \mathcal{X}$ yields a homomorphism $\mathcal{O}_{\mathcal{X}, r_{\mathcal{X}}(x)} \rightarrow k(x)^\circ$. In particular,

$$(1.2.2) \quad |f|_x \leq 1 \text{ for all } f \in \mathcal{O}_{\mathcal{X}, r_{\mathcal{X}}(x)}.$$

It is well-known that $r_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_\circ$ is anti-continuous, that is, for any open set U of \mathcal{X}_\circ , $r_{\mathcal{X}}^{-1}(U)$ is closed (cf. [2, Section 2.4]). Let Y be another proper algebraic scheme over k and $\mu : Y \rightarrow X$ a morphism over k . Let $\mathcal{Y} \rightarrow \text{Spec}(k^\circ)$ be a proper and flat scheme over $\text{Spec}(k^\circ)$ such that the generic fiber of $\mathcal{Y} \rightarrow \text{Spec}(k^\circ)$ is Y and there is a morphism $\tilde{\mu} : \mathcal{Y} \rightarrow \mathcal{X}$ over $\text{Spec}(k^\circ)$ as an extension of μ . It is easy to see that the following diagram is commutative:

$$(1.2.3) \quad \begin{array}{ccc} Y^{\text{an}} & \xrightarrow{r_{\mathcal{Y}}} & \mathcal{Y}_\circ \\ \mu^{\text{an}} \downarrow & \square & \downarrow \tilde{\mu} \\ X^{\text{an}} & \xrightarrow{r_{\mathcal{X}}} & \mathcal{X}_\circ \end{array}$$

1.3. Miscellaneous lemmas. In this subsection, we prove seven lemmas, which are non-trivial and indispensable for other subsections.

Lemma 1.3.1. *Let V be a vector space over \mathbb{Q} and W a subspace of V over \mathbb{Q} . Let $x_1, \dots, x_r \in V$ and $a_1, \dots, a_r \in \mathbb{R}$ such that a_1, \dots, a_r are linearly independent over \mathbb{Q} . If $a_1 x_1 + \dots + a_r x_r \in W \otimes_{\mathbb{Q}} \mathbb{R}$, then $x_1, \dots, x_r \in W$.*

Proof. We set $H := \{\phi \in \text{Hom}_{\mathbb{Q}}(V, \mathbb{Q}) \mid \phi|_W = 0\} = \text{Hom}_{\mathbb{Q}}(V/W, \mathbb{Q})$. For $\phi \in H$, the natural extension to $\text{Hom}_{\mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R})$ is denoted by $\phi_{\mathbb{R}}$. As $a_1 x_1 + \dots + a_r x_r \in W \otimes_{\mathbb{Q}} \mathbb{R}$, we have

$$0 = \phi_{\mathbb{R}}(a_1 x_1 + \dots + a_r x_r) = a_1 \phi(x_1) + \dots + a_r \phi(x_r)$$

for all $\phi \in H$. Thus $\phi(x_1) = \dots = \phi(x_r) = 0$ for all $\phi \in H$ because a_1, \dots, a_r are linearly independent over \mathbb{Q} and $\phi(x_1), \dots, \phi(x_r) \in \mathbb{Q}$. Therefore, $x_1, \dots, x_r \in W$. \square

Lemma 1.3.2. *Let V be a vector space over \mathbb{R} . Let $x_1, \dots, x_r \in V$, $a_1, \dots, a_r \in \mathbb{R}_{>0}$, $x := a_1 x_1 + \dots + a_r x_r$ and $\phi_1, \dots, \phi_m \in \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$. If $\phi_l(x) > 0$ for all $l = 1, \dots, m$, then there are $x'_1, \dots, x'_r \in \mathbb{Q}_{>0} x_1 + \dots + \mathbb{Q}_{>0} x_r$ and $a'_1, \dots, a'_r \in \mathbb{R}_{>0}$ such that $x = a'_1 x'_1 + \dots + a'_r x'_r$ and $\phi_l(x'_i) > 0$ for all $i = 1, \dots, r$ and $l = 1, \dots, m$.*

Proof. Let us begin with the following claim:

Claim 1.3.2.1. *Let $y_1, \dots, y_s \in V$, $b_1, \dots, b_s \in \mathbb{R}_{>0}$ and $\psi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$. If $\psi(y_i) > 0$ for $i = 1, \dots, s-1$ and $\psi(b_1 y_1 + \dots + b_s y_s) > 0$, then there are $c_1, \dots, c_{s-1} \in \mathbb{Q}_{>0}$ such that $b_i - c_i b_s > 0$ for $i = 1, \dots, s-1$ and*

$$\psi(c_1 y_1 + \dots + c_{s-1} y_{s-1} + y_s) > 0.$$

Proof. As $b_1\psi(y_1) + \cdots + b_{s-1}\psi(y_{s-1}) + b_s\psi(y_s) > 0$, we have

$$-\psi(y_s) < (b_1/b_s)\psi(y_1) + \cdots + (b_{s-1}/b_s)\psi(y_{s-1}).$$

Therefore, we can find $c_1, \dots, c_{s-1} \in \mathbb{Q}_{>0}$ such that

$$-\psi(y_s) < c_1\psi(y_1) + \cdots + c_{s-1}\psi(y_{s-1}) \quad \text{and} \quad c_i < b_i/b_s$$

for $i = 1, \dots, s-1$, as required. \square

First we consider the case where $m = 1$. We set $I = \{i \mid \phi_1(x_i) \leq 0\}$ and $J = \{i \mid \phi_1(x_i) > 0\}$. We prove the lemma for $m = 1$ by induction on $\#(I)$. As

$$\phi_1(x) = a_1\phi_1(x_1) + \cdots + a_r\phi_1(x_r) > 0,$$

we have $J \neq \emptyset$. Clearly we may assume that $I \neq \emptyset$. Choose $s \in I$. Note that

$$\phi_1 \left(\sum_{j \in J} a_j x_j + a_s x_s \right) > 0,$$

so that, applying Claim 1.3.2.1 to $\{x_j\}_{j \in J} \cup \{x_s\}$, $\{a_j\}_{j \in J} \cup \{a_s\}$ and ϕ_1 , we can find $c_j \in \mathbb{Q}_{>0}$ ($j \in J$) such that $a_j - c_j a_s > 0$ for $j \in J$ and

$$\phi_1 \left(\sum_{j \in J} c_j x_j + x_s \right) > 0.$$

Note that

$$x = \sum_{j \in J} (a_j - c_j a_s) x_j + a_s \left(\sum_{j \in J} c_j x_j + x_s \right) + \sum_{i \in I \setminus \{s\}} a_i x_i.$$

Thus, by the induction hypothesis, the assertion of the lemma for $m = 1$ follows.

In general, we prove the lemma induction on m . The previous observation shows that it holds for $m = 1$. By hypothesis of induction, there are $x'_1, \dots, x'_r \in \mathbb{Q}_{>0}x_1 + \cdots + \mathbb{Q}_{>0}x_r$ and $a'_1, \dots, a'_r \in \mathbb{R}_{>0}$ such that $x = a'_1x'_1 + \cdots + a'_rx'_r$ and $\phi_l(x'_i) > 0$ for all $i = 1, \dots, r$ and $l = 1, \dots, m-1$. Moreover, by the case where $m = 1$, we can find $x''_1, \dots, x''_r \in \mathbb{Q}_{>0}x'_1 + \cdots + \mathbb{Q}_{>0}x'_r$ and $a''_1, \dots, a''_r \in \mathbb{R}_{>0}$ such that $x = a''_1x''_1 + \cdots + a''_rx''_r$ and $\phi_m(x''_i) > 0$ for all $i = 1, \dots, r$. Note that ϕ_l ($l = 1, \dots, m-1$) is positive on $\mathbb{Q}_{>0}x'_1 + \cdots + \mathbb{Q}_{>0}x'_r$ and

$$\mathbb{Q}_{>0}x'_1 + \cdots + \mathbb{Q}_{>0}x'_r \subseteq \mathbb{Q}_{>0}x_1 + \cdots + \mathbb{Q}_{>0}x_r.$$

Thus the assertion follows. \square

Lemma 1.3.3. *Let M be a finitely generated \mathbb{Z} -module and let $\|\cdot\|$ and $\|\cdot\|'$ be norms of $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let M' be a submodule of M such that M/M' is a torsion group, so that $M_{\mathbb{R}} = M'_{\mathbb{R}} (= M' \otimes_{\mathbb{Z}} \mathbb{R})$. If $\|\cdot\| \leq \|\cdot\|'$, then*

$$(1.3.3.1) \quad \hat{\chi}(M', \|\cdot\|') \leq \hat{\chi}(M, \|\cdot\|).$$

Moreover, for $\lambda \in \mathbb{R}_{\geq 0}$, the following formulae hold:

$$(1.3.3.2) \quad \hat{h}^0(M, \exp(-\lambda)\|\cdot\|) \leq \hat{h}^0(M, \|\cdot\|) + \lambda \operatorname{rk} M + \log(3) \operatorname{rk} M,$$

$$(1.3.3.3) \quad \hat{\chi}(M, \exp(-\lambda)\|\cdot\|) = \hat{\chi}(M, \|\cdot\|) + \lambda \operatorname{rk} M,$$

$$(1.3.3.4) \quad \hat{h}^0(M, \|\cdot\|) \leq \hat{h}^0(M', \|\cdot\|) + \log \#(M/M') + \log(6) \operatorname{rk} M,$$

$$(1.3.3.5) \quad \hat{\chi}(M, \|\cdot\|) = \hat{\chi}(M', \|\cdot\|) + \log \#(M/M').$$

Proof. (1.3.3.1) is obvious. (1.3.3.2) follows from [24, Lemma 1.2.2]. (1.3.3.3) is also obvious because $B(M, \exp(-\lambda)\|\cdot\|) = \exp(\lambda)B(M, \|\cdot\|)$.

Let us consider (1.3.3.4). Let $\pi : M \rightarrow M/M'$ be the canonical homomorphism. Let us choose $x_1, \dots, x_N \in M$ with the following properties:

- (i) $\|x_i\| \leq 1$ for all $i = 1, \dots, N$.
- (ii) $\pi(x_i) \neq \pi(x_j)$ for $i \neq j$.
- (iii) For any $x \in M$ with $\|x\| \leq 1$, there is x_i with $\pi(x) = \pi(x_i)$.

Then we can see that

$$\{x \in M \mid \|x\| \leq 1\} \subseteq \{x_i + x' \mid x' \in M' \text{ and } \|x'\| \leq 2\},$$

and hence,

$$\hat{h}^0(M, \|\cdot\|) \leq \hat{h}^0(M', (1/2)\|\cdot\|) + \log \#(M/M').$$

Therefore, (1.3.3.4) follows from (1.3.3.2).

For (1.3.3.5), let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M'/M'_{tor} & \longrightarrow & M/M_{tor} & \longrightarrow & (M/M_{tor})/(M'/M'_{tor}) & \longrightarrow & 0, \end{array}$$

which show that we may assume that M is torsion free. Let $\omega_1, \dots, \omega_r$ be a free basis of M such that $a_1\omega_1, \dots, a_r\omega_r$ form a free basis of M' for some $a_1, \dots, a_r \in \mathbb{Z}_{>0}$. Then

$$\text{vol}(M_{\mathbb{R}}/M') = a_1 \cdots a_r \text{vol}(M_{\mathbb{R}}/M).$$

Thus (1.3.3.5) follows because $\#(M'/M) = a_1 \cdots a_r$. \square

Lemma 1.3.4. *Let A and M be \mathbb{Z} -modules and let $f : M^n \rightarrow A$ be a multi-linear map, that is,*

$$f(x_1, \dots, x_i - x'_i, \dots, x_n) = f(x_1, \dots, x_i, \dots, x_n) - f(x_1, \dots, x'_i, \dots, x_n)$$

for all $i = 1, \dots, n$ and $x_1, \dots, x_i, x'_i, \dots, x_n \in M$. Then, for $x_1, \dots, x_n, x'_1, \dots, x'_n \in M$,

$$f(x'_1, \dots, x'_n) = f(x_1, \dots, x_n) + \sum_{i=1}^n f(x'_1, \dots, x'_{i-1}, \delta_i, x_{i+1}, \dots, x_n),$$

where $\delta_i = x'_i - x_i$.

Proof. We prove it by induction on n . In the case where $n = 1$, the above means that $f(x'_1) = f(x_1) + f(x'_1 - x_1)$, which is obvious. In general, using the induction hypothesis, we have

$$f(x'_1, \dots, x'_n) = f(x'_1, \dots, x'_{n-1}, x_n) + f(x'_1, \dots, x'_{n-1}, \delta_{n-1})$$

and

$$\begin{aligned} f(x'_1, \dots, x'_{n-1}, x_n) &= f(x_1, \dots, x_{n-1}, x_n) \\ &\quad + \sum_{i=1}^{n-1} f(x'_1, \dots, x'_{i-1}, \delta_i, x_{i+1}, \dots, x_{n-1}, x_n). \end{aligned}$$

Thus we have the assertion. \square

Lemma 1.3.5. *Let V be a vector space over \mathbb{R} and W a subspace of V over \mathbb{R} . Let*

$$\langle \rangle : W \times V^{l-1} \rightarrow \mathbb{R}$$

be a multi-linear map over \mathbb{R} . We assume that $\langle \rangle$ is symmetric, that is,

$$\langle x_1, x_2, \dots, x_l \rangle = \langle x_2, x_1, \dots, x_l \rangle$$

for all $x_1, x_2 \in W$ and $x_3, \dots, x_l \in V$ and

$$\langle x_1, \dots, x_i, \dots, x_j, \dots, x_l \rangle = \langle x_1, \dots, x_j, \dots, x_i, \dots, x_l \rangle$$

for all $x_1 \in W$, $x_2, \dots, x_l \in V$ and $2 \leq i < j \leq l$. For simplicity, $\langle x_1, \dots, x_l \rangle$ is denoted by $\langle x_1 \cdots x_l \rangle$ or $\langle \prod_i x_i \rangle$. Then

$$\begin{aligned} & \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} (a_i + b_i) \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ &= \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} b_i \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle + \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} a_i \cdot \prod_{j \in \{1, \dots, l\} \setminus I} (x_j + b_j) \right\rangle \end{aligned}$$

holds for $a_1, \dots, a_l, b_1, \dots, b_l \in W$ and $x_1, \dots, x_l \in V$.

Proof. If we set

$$\begin{aligned} A := \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} & \left\langle \prod_{i \in I} (a_i + b_i) \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ & - \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} b_i \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle, \end{aligned}$$

then

$$\begin{aligned} A &= \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \sum_{L \subseteq I} \left\langle \prod_{i \in L} a_i \cdot \prod_{i' \in I \setminus L} b_{i'} \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ &\quad - \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in I} b_i \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ &= \sum_{\emptyset \neq I \subseteq \{1, \dots, l\}} \sum_{\emptyset \neq L \subseteq I} \left\langle \prod_{i \in L} a_i \cdot \prod_{i' \in I \setminus L} b_{i'} \cdot \prod_{j \in \{1, \dots, l\} \setminus I} x_j \right\rangle \\ &= \sum_{\emptyset \neq L \subseteq \{1, \dots, l\}} \sum_{M \amalg M' = \{1, \dots, l\} \setminus L} \left\langle \prod_{i \in L} a_i \cdot \prod_{m \in M} b_m \cdot \prod_{m' \in M'} x_{m'} \right\rangle \\ &= \sum_{\emptyset \neq L \subseteq \{1, \dots, l\}} \left\langle \prod_{i \in L} a_i \cdot \prod_{j \in \{1, \dots, l\} \setminus L} (x_j + b_j) \right\rangle, \end{aligned}$$

as desired. \square

Lemma 1.3.6. *Let S be a connected Dedekind scheme and k the rational function field of S . Let X be a projective variety over k . Then we have the following:*

- (1) *There exists a model of X over S (cf. Conventions and terminology 0.5.5).*

- (2) Let J be an invertible fractional ideal sheaf on X . Then there are a model \mathcal{X} of X over S and an invertible fractional ideal sheaf \mathcal{J} on \mathcal{X} such that $\mathcal{J} \cap X = J$.
- (3) Let D be an \mathbb{R} -Cartier divisor on X . Then there are a model \mathcal{X} of X over S and an \mathbb{R} -Cartier divisor \mathcal{D} on \mathcal{X} such that $\mathcal{D} \cap X = D$.

Proof. (1) As X is projective over k , there is a closed embedding $\iota : X \hookrightarrow \mathbb{P}_k^N$. Let \mathcal{X} be the closure of $\iota(X)$ in \mathbb{P}_S^N . Then \mathcal{X} is integral, projective and flat over S because S is a connected Dedekind scheme.

(2) Let \mathcal{X}' be a model of X over S . Then we can find a non-empty open set U of S and an invertible fractional ideal sheaf \mathcal{J}'_U on \mathcal{X}'_U such that $\mathcal{J}'_U \cap X = J$. Therefore, as \mathcal{X} is noetherian, by using the extension theorem of coherent sheaves (cf. [14, Chapter II, Exercise 5.15]), we have a fractional ideal sheaf \mathcal{J}' on \mathcal{X}' such that $\mathcal{J}' \cap \mathcal{X}'_U = \mathcal{J}'_U$. Let $\pi : \mathcal{X} = \text{Proj}(\bigoplus_{m=0}^{\infty} \mathcal{J}'^m) \rightarrow \mathcal{X}'$ be the blowing-up by the fractional ideal sheaf \mathcal{J}' . Then, as $\mathcal{J} := \mathcal{J}' \mathcal{O}_{\mathcal{X}}$ is invertible, the assertion of (2) follows.

(3) is a consequence of (2). \square

Lemma 1.3.7. *Let k be a field and v a valuation of k . Let k_v be the completion of k with respect to v . By abuse of notation, the unique extension of v to k_v is also denoted by v . Then we have the following:*

- (1) *We assume that v is discrete. Let X be a projective and geometrically integral variety over k and let \mathcal{X} be a model of X over $\text{Spec}(k^\circ)$. We set*

$$X_v := X \times_{\text{Spec}(k)} \text{Spec}(k_v) \quad \text{and} \quad \mathcal{X}_v := \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k_v^\circ).$$

Let $\pi : \mathcal{X}_v \rightarrow \mathcal{X}$ be the projection, and let $(\mathcal{X}_v)_\circ$ and \mathcal{X}_\circ be the central fibers of

$$\mathcal{X}_v \rightarrow \text{Spec}(k_v^\circ) \quad \text{and} \quad \mathcal{X} \rightarrow \text{Spec}(k^\circ),$$

respectively (cf. Conventions and terminology 0.5.2). If we choose $\xi_v \in (\mathcal{X}_v)_\circ$ and $\xi \in \mathcal{X}_\circ$ with $\pi(\xi_v) = \xi$, then we have the following:

- (1.1) \mathcal{X}_v is a model of X_v over $\text{Spec}(k_v^\circ)$.
- (1.2) $\mathcal{O}_{\mathcal{X}, \xi}$ is regular if and only if $\mathcal{O}_{\mathcal{X}_v, \xi_v}$ is regular.
- (1.3) We assume that k° is excellent. Then $\mathcal{O}_{\mathcal{X}, \xi}$ is normal if and only if $\mathcal{O}_{\mathcal{X}_v, \xi_v}$ is normal.

- (2) *Let A be a k -algebra and $A_v := A \otimes_k k_v$. Let $x = |\cdot|_x$ and $x' = |\cdot|_{x'}$ be seminorms of A_v . If $|a \otimes 1|_x = |a \otimes 1|_{x'}$ for all $a \in A$, then $x = x'$.*

Proof. (1) We need to see that \mathcal{X}_v is integral. As $\mathcal{X}_v \rightarrow \text{Spec}(k_v^\circ)$ is flat and the generic fiber of $\mathcal{X}_v \rightarrow \text{Spec}(k_v^\circ)$ is integral, (1.1) follows from [20, Lemma 4.2].

Before starting the proofs of (1.2) and (1.3), let us see $m_{\xi_v} = m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v}$. Let $\kappa(\xi)$ be the residue field at ξ . Here we consider an exact sequence

$$\kappa(\xi) \otimes_{k^\circ} k_v^{\circ\circ} \xrightarrow{\alpha} \kappa(\xi) \otimes_{k^\circ} k_v^\circ \longrightarrow \kappa(\xi) \otimes_{k^\circ} (k_v^\circ / k_v^{\circ\circ}) \longrightarrow 0$$

induced by $0 \rightarrow k_v^{\circ\circ} \rightarrow k_v^\circ \rightarrow k_v^\circ / k_v^{\circ\circ} \rightarrow 0$. Note that $\alpha = 0$ because $\alpha(a \otimes \varpi b) = a \otimes \varpi b = \varpi a \otimes b = 0$ for $a \in \kappa(\xi)$ and $b \in k_v^\circ$. Therefore,

$$\kappa(\xi) \otimes_{k^\circ} k_v^\circ \simeq \kappa(\xi) \otimes_{k^\circ} (k_v^\circ / k_v^{\circ\circ}) \simeq \kappa(\xi) \otimes_{k^\circ} (k^\circ / k^{\circ\circ}) \simeq \kappa(\xi).$$

On the other hand, performing $\otimes_{\mathcal{O}_{\mathcal{X}, \xi}} \mathcal{O}_{\mathcal{X}_v, \xi_v}$ to the exact sequence $0 \rightarrow m_\xi \rightarrow \mathcal{O}_{\mathcal{X}, \xi} \rightarrow \kappa(\xi) \rightarrow 0$, we obtain

$$0 \rightarrow m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v} \rightarrow \mathcal{O}_{\mathcal{X}_v, \xi_v} \rightarrow \kappa(\xi) \otimes_{\mathcal{O}_{\mathcal{X}, \xi}} \mathcal{O}_{\mathcal{X}_v, \xi_v} \rightarrow 0.$$

As π induces the isomorphism $(\mathcal{X}_v)_\circ \xrightarrow{\sim} \mathcal{X}_\circ$, $\pi^{-1}(\xi) = \{\xi_v\}$, so that $\mathcal{O}_{\mathcal{X}_v, \xi_v} = \mathcal{O}_{\mathcal{X}, \xi} \otimes_{k^\circ} k_v^\circ$. Therefore,

$$\kappa(\xi) \otimes_{\mathcal{O}_{\mathcal{X}, \xi}} \mathcal{O}_{\mathcal{X}_v, \xi_v} = \kappa(\xi) \otimes_{\mathcal{O}_{\mathcal{X}, \xi}} (\mathcal{O}_{\mathcal{X}, \xi} \otimes_{k^\circ} k_v^\circ) \simeq \kappa(\xi) \otimes_{k^\circ} k_v^\circ \simeq \kappa(\xi),$$

which shows that $m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v}$ is the maximal ideal of $\mathcal{O}_{\mathcal{X}_v, \xi_v}$, and hence $m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v} = m_{\xi_v}$.

Since $\mathcal{O}_{\mathcal{X}, \xi} \subseteq \mathcal{O}_{\mathcal{X}_v, \xi_v} \subseteq \widehat{\mathcal{O}}_{\mathcal{X}, \xi}$ and $m_\xi \mathcal{O}_{\mathcal{X}_v, \xi_v} = m_{\xi_v}$, by [19, Chapter 1, Theorem 3.16], we have $\widehat{\mathcal{O}}_{\mathcal{X}_v, \xi_v} \simeq \widehat{\mathcal{O}}_{\mathcal{X}, \xi}$. Thus (1.2) follows from [1, Proposition 11.24]. Further, (1.3) follows from [11, VI, 7.8.3, (v)].

(2) For $\alpha \in A_v$, we set $\alpha = a_1 \otimes \lambda_1 + \cdots + a_r \otimes \lambda_r$, where $a_1, \dots, a_r \in A$ and $\lambda_1, \dots, \lambda_r \in k_v$. Then we can find sequences $\{\lambda_{1,n}\}_{n=1}^\infty, \dots, \{\lambda_{r,n}\}_{n=1}^\infty$ in k such that $\lambda_i = \lim_{n \rightarrow \infty} \lambda_{i,n}$ for $i = 1, \dots, r$. Here we set

$$\alpha_n = a_1 \otimes \lambda_{1,n} + \cdots + a_r \otimes \lambda_{r,n} = (\lambda_{1,n} a_1 + \cdots + \lambda_{r,n} a_r) \otimes 1.$$

Then,

$$\begin{aligned} |\alpha_n|_x - |\alpha|_x &\leq |\alpha_n - \alpha|_x = |a_1 \otimes (\lambda_{1,n} - \lambda_1) + \cdots + a_r \otimes (\lambda_{r,n} - \lambda_r)|_x \\ &\leq |a_1 \otimes (\lambda_{1,n} - \lambda_1)|_x + \cdots + |a_r \otimes (\lambda_{r,n} - \lambda_r)|_x \\ &= |(a_1 \otimes 1) \cdot (1 \otimes (\lambda_{1,n} - \lambda_1))|_x + \cdots + |(a_r \otimes 1) \cdot (1 \otimes (\lambda_{r,n} - \lambda_r))|_x \\ &= |a_1 \otimes 1|_x v(\lambda_{1,n} - \lambda_1) + \cdots + |a_r \otimes 1|_x v(\lambda_{r,n} - \lambda_r), \end{aligned}$$

and hence $\lim_{n \rightarrow \infty} |\alpha_n|_x = |\alpha|_x$. In the same way, $\lim_{n \rightarrow \infty} |\alpha_n|_{x'} = |\alpha|_{x'}$. On the other hand, by our assumption,

$$|\alpha_n|_x = |(\lambda_{1,n} a_1 + \cdots + \lambda_{r,n} a_r) \otimes 1|_x = |(\lambda_{1,n} a_1 + \cdots + \lambda_{r,n} a_r) \otimes 1|_{x'} = |\alpha_n|_{x'}$$

for all $n \geq 1$. Therefore, $|\alpha|_x = |\alpha|_{x'}$, and hence $x = x'$. \square

2. ADELIC \mathbb{R} -CARTIER DIVISORS OVER A DISCRETE VALUATION FIELD

In this section, we introduce an adelic \mathbb{R} -Cartier divisor on a projective variety over a discrete valuation field and study their basic properties. Roughly speaking, an adelic \mathbb{R} -Cartier divisor is a pair of an \mathbb{R} -Cartier divisor and a Green function on the analytification of the given variety, which is an analogue of Arakelov divisors (i.e. arithmetic divisors) on an arithmetic variety.

Throughout this section, let k be a field and v a discrete valuation. We set

$$k^\circ := \{a \in k \mid v(a) \leq 1\} \quad \text{and} \quad k^{\circ\circ} := \{a \in k \mid v(a) < 1\}.$$

Let ϖ be a uniformizing parameter of v , that is, $k^{\circ\circ} = \varpi k^\circ$. Note that v might be trivial, so that we do not exclude the case where $\varpi = 0$. Let us begin with Green functions on analytic spaces over a complete discrete valuation field.

2.1. Green functions on analytic spaces over a discrete valuation field. We assume that v is complete. Let X be a projective and geometrically integral variety over k . Let $\text{Rat}(X)$ be the rational function field of X . Let $U = \text{Spec}(A)$ be an affine open set of X . Let $p \in U$ and $x = |\cdot|_x \in U^{\text{an}}$ such that $p_x \subseteq p$, where p_x is the associated prime of x (cf. Subsection 1.2). Then we have a natural extension $|\cdot|_x : A_p \rightarrow \mathbb{R}_{\geq 0}$ of $|\cdot|_x$ on A given by $|a/s|_x = |a|_x/|s|_x$ for $a \in A$ and $s \in A \setminus p$, which yields the group homomorphism $|\cdot|_x : A_p^\times \rightarrow \mathbb{R}_{>0}$. Thus we obtain a canonical extension $(A_p^\times)_{\mathbb{R}} \rightarrow \mathbb{R}_{>0}$, which is also denoted by $|\cdot|_x$ by abuse of notation. Let $f \in \text{Rat}(X)_{\mathbb{R}}^\times$ and $x \in U^{\text{an}} \setminus V_{\mathbb{R}}(f)^{\text{an}}$ (see Subsection 1.1 for the definitions of $\text{Rat}(X)_{\mathbb{R}}^\times$ and $V_{\mathbb{R}}(f)$). As $p_x \notin V_{\mathbb{R}}(f)$, we get $f \in (A_{p_x}^\times)_{\mathbb{R}}$, and hence

$|f(x)| \in \mathbb{R}_{>0}$. Therefore, we have a map $U^{\text{an}} \setminus V_{\mathbb{R}}(f)^{\text{an}} \rightarrow \mathbb{R}$ given by $x \mapsto \log|f(x)|^2$. We denote it by $\log|f|^2$. Clearly $\log|f|^2$ is continuous on $U^{\text{an}} \setminus V_{\mathbb{R}}(f)^{\text{an}}$.

Definition 2.1.1. Let D be an \mathbb{R} -Cartier divisor on X , that is, $D \in \text{Div}(X)_{\mathbb{R}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $X = \bigcup_{i=1}^N U_i$ be an affine open covering of X such that D is given by $f_i \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ on U_i . A continuous function $g : X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}} \rightarrow \mathbb{R}$ is called a D -Green function of C^0 -type on X^{an} if $g + \log|f_i|^2$ extends to a continuous function on U_i^{an} for each $i = 1, \dots, N$.

For example, for $f \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, $-\log|f|^2$ is $(f)_{\mathbb{R}}$ -Green function of C^0 -type on X^{an} , where $(f)_{\mathbb{R}}$ is the \mathbb{R} -principal divisor of f (cf. Subsection 1.1). We set

$$C_{\eta}^0(X^{\text{an}}) := \varinjlim_{U: \text{Zariski open set of } X} C^0(U^{\text{an}}).$$

The space of all Green functions forms a subspace of $C_{\eta}^0(X^{\text{an}})$ over \mathbb{R} . More precisely, we have the following proposition:

Proposition 2.1.2. *Let D and D' be \mathbb{R} -Cartier divisors on X . Let g be a D -Green function of C^0 -type on X^{an} and g' a D' -Green function of C^0 -type on X^{an} . Then we have the following:*

- (1) *For $a, b \in \mathbb{R}$, $ag + bg'$ is an $(aD + bD')$ -Green function of C^0 -type.*
- (2) *If $D = D'$, then $\max\{g, g'\}$ and $\min\{g, g'\}$ are D -Green functions of C^0 -type.*

Proof. (1) Let $X = \bigcup_{i=1}^N U_i$ be an affine open covering of X such that D and D' are given by f_i and f'_i on U_i , respectively. By our assumption, there are continuous functions φ and φ' on U_i^{an} such that $g = -\log|f_i|^2 + \varphi$ and $g' = -\log|f'_i|^2 + \varphi'$. Thus

$$ag + bg' = -\log|f_i^a f_i'^b|^2 + a\varphi + b\varphi'.$$

Note that $f_i^a f_i'^b$ is a local equation of $aD + bD'$ on U_i . Thus (1) follows.

(2) Note that

$$\max\{g, g'\} = -\log|f_i|^2 + \max\{\varphi, \varphi'\} \quad \text{and} \quad \min\{g, g'\} = -\log|f_i|^2 + \min\{\varphi, \varphi'\}$$

on $U_i^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Moreover, $\max\{\varphi, \varphi'\}$ and $\min\{\varphi, \varphi'\}$ are continuous on U_i^{an} , as required. \square

Next let us consider a norm arising from a Green function.

Proposition 2.1.3. *We assume that X is normal. We set*

$$H^0(X, D) := \{\phi \in \text{Rat}(X)^{\times} \mid (\phi) + D \geq 0\} \cup \{0\}.$$

Let g be a D -Green function of C^0 -type on X^{an} . Then we have the following:

- (1) *For $\phi \in H^0(X, D)$, $|\phi| \exp(-g/2)$ extends to a continuous function ϑ on X^{an} . We denote $\|\vartheta\|_{\text{sup}}$ by $\|\phi\|_g$.*
- (2) *The following formulae hold:*
 - (2.1) $\|a\phi\|_g = v(a)\|\phi\|_g$ for all $a \in k$ and $\phi \in H^0(X, D)$.
 - (2.2) $\|\phi_1 + \phi_2\|_g \leq \max\{\|\phi_1\|_g, \|\phi_2\|_g\}$ for all $\phi_1, \phi_2 \in H^0(X, D)$.

Proof. (1) Clearly we may assume that $\phi \neq 0$. Let $X = \bigcup_{i=1}^N \text{Spec}(A_i)$ be an affine open covering of X such that D is given by $h_i \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ on $\text{Spec}(A_i)$. Since $D + (\phi)$ is effective as a Weil divisor, $\text{ord}_{\Gamma}(\phi h_i) \geq 0$ for any prime divisor Γ on $\text{Spec}(A_i)$. Thus, by virtue of Hartogs' lemma for \mathbb{R} -rational functions (cf. Lemma 1.1.4), there are $u_1, \dots, u_r \in$

$A_i \setminus \{0\}$ and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ with $\phi h_i = u_1^{a_1} \cdots u_r^{a_r}$. In particular, $|\phi h_i| = |u_1|^{a_1} \cdots |u_r|^{a_r}$ is continuous on $\text{Spec}_k^{\text{an}}(A_i)$. On the other hand, there is a continuous function φ_i on $\text{Spec}_k^{\text{an}}(A_i)$ such that $g = -\log|h_i|^2 + \varphi_i$ on $\text{Spec}_k^{\text{an}}(A_i)$. Therefore,

$$|\phi| \exp(-g/2) = |\phi h_i| \exp(-\varphi_i/2)$$

is continuous on $\text{Spec}_k^{\text{an}}(A_i)$.

(2.1) is obvious. (2.2) is also obvious because $|\phi_1 + \phi_2| \leq \max\{|\phi_1|, |\phi_2|\}$ on some dense open set. \square

A pair $(\mathcal{X}, \mathcal{D})$ is called a *model of (X, D)* if \mathcal{X} is a model of X over $\text{Spec}(k^\circ)$ (cf. Conventions and terminology 0.5.5) and \mathcal{D} is an \mathbb{R} -Cartier divisor on \mathcal{X} with $\mathcal{D} \cap X = D$. The \mathbb{R} -Cartier divisor \mathcal{D} is often called a *model of D on \mathcal{X}* . For $x \in X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, let f be a local equation of \mathcal{D} at $\xi = r_{\mathcal{X}}(x)$, where $r_{\mathcal{X}}$ is the reduction map $X^{\text{an}} \rightarrow \mathcal{X}_\circ$ (cf. Subsection 1.2). As $p_x \in \text{Spec}(\mathcal{O}_{\mathcal{X}, \xi})$ and $f \in (\mathcal{O}_{\mathcal{X}, p_x}^\times)_{\mathbb{R}}$, we have $|f(x)| \neq 0$, so that we can define $g_{(\mathcal{X}, \mathcal{D})}(x)$ to be

$$g_{(\mathcal{X}, \mathcal{D})}(x) := -\log|f(x)|^2.$$

Let f' be another local equation of \mathcal{D} at ξ . Then there is $u \in (\mathcal{O}_{\mathcal{X}, \xi}^\times)_{\mathbb{R}}$ such that $f' = fu$, and hence $|f(x)| = |f'(x)|$ because $|u(x)| = 1$ (cf. (1.2.2)). Therefore, $g_{(\mathcal{X}, \mathcal{D})}(x)$ does not depend on the choice of f . Let us see the following proposition:

- Proposition 2.1.4.** (1) $g_{(\mathcal{X}, \mathcal{D})}$ is a D -Green function of C^0 -type on X^{an} .
 (2) Let Y be another projective and geometrically integral variety over k and let $v : Y \rightarrow X$ be a morphism over k . Let \mathcal{Y} be a model of Y such that there is a morphism $\hat{v} : \mathcal{Y} \rightarrow \mathcal{X}$ over $\text{Spec}(k^\circ)$ as an extension of v . Then $g_{(\mathcal{Y}, \hat{v}^*(\mathcal{D}))} = v^{\text{an}} \circ g_{(\mathcal{X}, \mathcal{D})}$ on $Y^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(v^*(D))^{\text{an}}$.
 (3) Let $\tilde{\mu} : \mathcal{X}' \rightarrow \mathcal{X}$ be the normalization of \mathcal{X} . Let X' the generic fiber of $\mathcal{X}' \rightarrow \text{Spec}(k^\circ)$ and $\mu : X' \rightarrow X$ the induced morphism (note that X' is normal). We assume that the associated \mathbb{R} -Weil divisor of $\mu^*(D)$ is effective. Then the associated \mathbb{R} -Weil divisor of $\tilde{\mu}^*(\mathcal{D})$ is effective if and only if $g_{(\mathcal{X}, \mathcal{D})} \geq 0$ on $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$.

Proof. (1) Let $\mathcal{X} = \bigcup_{i=1}^N \text{Spec}(\mathcal{A}_i)$ be an affine open covering of \mathcal{X} such that we have a local equation $f_i \in \text{Rat}(X)_{\mathbb{R}}^\times$ of \mathcal{D} on $\mathcal{U}_i := \text{Spec}(\mathcal{A}_i)$, where \mathcal{A}_i is a k° -algebra for each $i = 1, \dots, N$. We set $C_i = r_{\mathcal{X}}^{-1}(\mathcal{U}_i \cap \mathcal{X}_\circ)$. Then C_i is closed (cf. [2, Section 2.4]) and $\bigcup_{i=1}^N C_i = X^{\text{an}}$.

First let us see that $g_{(\mathcal{X}, \mathcal{D})} : X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}} \rightarrow \mathbb{R}$ is continuous. By our construction, $g_{(\mathcal{X}, \mathcal{D})}(x) = -\log|f_i(x)|^2$ for $x \in C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Thus $g_{(\mathcal{X}, \mathcal{D})}$ is continuous on $C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Let Z be a closed subset in $\mathbb{R}_{\geq 0}$. As

$$\left(g_{(\mathcal{X}, \mathcal{D})} \Big|_{C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}} \right)^{-1}(Z) = g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}),$$

$g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}})$ is closed in $C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, and hence

$$g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}})$$

is closed in $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Note that

$$\begin{aligned} \bigcup_{i=1}^N \left(g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}) \right) \\ = g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z) \cap \bigcup_{i=1}^N (C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}) = g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z). \end{aligned}$$

Thus $g_{(\mathcal{X}, \mathcal{D})}^{-1}(Z)$ is closed in $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, so that $g_{(\mathcal{X}, \mathcal{D})}$ is continuous on $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$.

Since f_i is a local equation of D on $U_i = \mathcal{U}_i \cap X$, in order to see that $g_{(\mathcal{X}, \mathcal{D})}$ is a D -Green function of C^0 -type, it is sufficient to see that $\psi = g_{(\mathcal{X}, \mathcal{D})} + \log|f_i|^2$ extends to a continuous function on U_i^{an} , which is obvious because $\psi = 0$ on U_i^{an} .

(2) First note that $v(\text{Supp}_{\mathbb{R}}(v^*(D))) \subseteq \text{Supp}_{\mathbb{R}}(D)$, so that

$$(v^{\text{an}})^{-1}(X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}) \subseteq Y^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(v^*(D))^{\text{an}}.$$

We set $C'_i = r_{\mathcal{Y}}^{-1}(\tilde{v}^{-1}(\mathcal{U}_i) \cap \mathcal{Y}_o)$. Let $y \in C'_i \setminus \text{Supp}(v^*(D))^{\text{an}}$, $\xi' = r_{\mathcal{Y}}(y)$ and $\xi = r_{\mathcal{X}}(v^{\text{an}}(y))$. Note that $\xi = \tilde{v}(\xi') \in \mathcal{U}_i \cap \mathcal{X}_o$ by (1.2.2). Then, as $\tilde{v}^*(f_i)$ is a local equation of $\tilde{v}^*(\mathcal{D})$ at ξ' ,

$$g_{(\mathcal{Y}, \tilde{v}^*(\mathcal{D}))}(y) = -\log|\tilde{v}^*(f_i)|_y^2 = -\log|f_i|_{v^{\text{an}}(y)}^2 = g_{(\mathcal{X}, \mathcal{D})}(v^{\text{an}}(y)),$$

as required.

(3) By virtue of (2), $g_{(\mathcal{X}', \tilde{\mu}^*(\mathcal{D}))} = g_{(\mathcal{X}, \mathcal{D})} \circ \mu^{\text{an}}$ on $(X')^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(\mu^*(D))^{\text{an}}$. Moreover, $\mu^{\text{an}} : (X')^{\text{an}} \rightarrow X^{\text{an}}$ is surjective by [2, Proposition 3.4.6]. Thus we may assume that \mathcal{X} is normal.

First we assume that \mathcal{D} is effective as a Weil divisor. Then, $\text{ord}_{\Gamma}(f_i) \geq 0$ for any prime divisor Γ on \mathcal{U}_i . Thus, by Hartogs' lemma for \mathbb{R} -rational functions (cf. Lemma 1.1.4), there are $h_1, \dots, h_r \in \mathcal{A}_i \setminus \{0\}$ and $a_1, \dots, a_r \in \mathbb{R}_{>0}$ with $f_i = h_1^{a_1} \cdots h_r^{a_r}$. Note that $|h_j|_x \leq 1$ for $j = 1, \dots, r$ and $x \in C_i \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, and hence $|f_i|_x \leq 1$, as required.

Next we assume that $g_{(\mathcal{X}, \mathcal{D})} \geq 0$ on $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$. Let us see that \mathcal{D} is effective as a Weil divisor. If v is trivial, then $\mathcal{D} = D$ is effective, so that we may assume that v is non-trivial. It is sufficient to show that the coefficient of \mathcal{D}_W with respect to a vertical prime divisor Γ is non-negative. Let us consider a multiplicative seminorm $x_{\Gamma} = |\cdot|_{x_{\Gamma}}$ given by

$$|f|_{x_{\Gamma}} := v(\varpi)^{\text{ord}_{\Gamma}(f)/\text{ord}_{\Gamma}(\varpi)}$$

for $f \in \text{Rat}(X)$, where ϖ is a uniformizing parameter of k^o . As $x_{\Gamma} \in X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, if $x_{\Gamma} \in C_i$, then

$$0 \leq g_{(\mathcal{X}, \mathcal{D})}(x_{\Gamma}) = -\log|f_i|_{x_{\Gamma}}^2 = -2\log v(\varpi) \frac{\text{ord}_{\Gamma}(f_i)}{\text{ord}_{\Gamma}(\varpi)},$$

and hence $\text{ord}_{\Gamma}(f_i) \geq 0$, as desired. \square

Here we discuss a more sophisticated maximum problem of Green functions on a smooth projective curve than (2) in Proposition 2.1.2.

Proposition 2.1.5. *We assume that X is a smooth projective curve over k . Let D_1, \dots, D_r be \mathbb{R} -Cartier divisors on X and let $D := \max\{D_1, \dots, D_r\}$ (Conventions and terminology 0.5.8). For each $i = 1, \dots, r$, let g_i be a D_i -Green function of C^0 -type on X^{an} . We set*

$$g := \max\{g_1, \dots, g_r\}$$

on $X^{\text{an}} \setminus (\text{Supp}_{\mathbb{R}}(D_1)^{\text{an}} \cup \dots \cup \text{Supp}_{\mathbb{R}}(D_r)^{\text{an}})$. Then g extends to a continuous function on $X^{\text{an}} \setminus \text{Supp}_{\mathbb{R}}(D)^{\text{an}}$, and g yields a D -Green function of C^0 -type on X^{an} , which is also denoted by

$$\max\{g_1, \dots, g_r\}$$

by abuse of notation.

Proof. Let x_1, \dots, x_N be closed points of X such that, for each $i = 1, \dots, r$,

$$D_i = a_{i1}x_1 + \dots + a_{iN}x_N$$

for some $a_{ij} \in \mathbb{R}$. If we set $a_j = \max\{a_{1j}, \dots, a_{rj}\}$ for $j = 1, \dots, N$, then

$$D = a_1x_1 + \dots + a_Nx_N.$$

Let \tilde{x}_j be a unique point of X^{an} such that $\{\tilde{x}_j\} = \{x_j\}^{\text{an}}$.

Claim 2.1.5.1. *If $D = 0$, then g is continuous on X^{an} .*

Proof. Clearly g is continuous on $X^{\text{an}} \setminus \{\tilde{x}_1, \dots, \tilde{x}_N\}$, so that we need to show that g is continuous at each \tilde{x}_j . Let us choose an affine open set U of X such that

$$\{x_1, \dots, x_N\} \cap U = \{x_j\}.$$

As $a_{ij} \leq 0$, we can see that g_i is upper-semicontinuous on U^{an} . Moreover, g_i is continuous on U^{an} if $a_{ij} = 0$. We set

$$I := \{i = 1, \dots, r \mid a_{ij} = 0\} \quad \text{and} \quad I' := \{i = 1, \dots, r \mid a_{ij} < 0\}.$$

Note that $I \neq \emptyset$ because $\max\{a_{1j}, \dots, a_{rj}\} = a_j = 0$, so that we choose $i_0 \in I$. Here we set

$$V := \{x \in U^{\text{an}} \mid g_{i_0}(x) > g_{i_0}(\tilde{x}_j) - 1 \text{ and } g_{i'}(x) < g_{i_0}(\tilde{x}_j) - 1 \text{ for all } i' \in I'\}.$$

Then V is an open set and $\tilde{x}_j \in V$. Further

$$\max\{g_i(x) \mid i \in I\} \geq g_{i_0}(x) > g_{i_0}(\tilde{x}_j) - 1 > \max\{g_{i'}(x) \mid i' \in I'\}$$

on $V \setminus \{\tilde{x}_j\}$, and hence $g(x) = \max\{g_i(x) \mid i \in I\}$ on $V \setminus \{\tilde{x}_j\}$. Therefore, the claim follows because g_i is continuous on U^{an} for $i \in I$. \square

Let g' be a D -Green function of C^0 -type on X^{an} . It is sufficient to see that $g - g'$ extends to a continuous function on X^{an} . Clearly $g - g'$ is continuous on $X^{\text{an}} \setminus \{\tilde{x}_1, \dots, \tilde{x}_N\}$ and

$$g - g' = \max\{g_1 - g', \dots, g_r - g'\}$$

on $X^{\text{an}} \setminus \{\tilde{x}_1, \dots, \tilde{x}_N\}$. Note that $g_i - g'$ is a $(D_i - D)$ -Green function of C^0 -type on X^{an} and $\max\{D_1 - D, \dots, D_r - D\} = 0$, so that the assertion follows from the above claim. \square

Finally let us consider a Green function of $(C^0 \cap \text{PSH})$ -type, which is a counterpart of a semipositive metric.

Definition 2.1.6. Let g be a D -Green function of C^0 -type on X^{an} . We say g is of $(C^0 \cap \text{PSH})$ -type if D is nef and there is a sequence $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^{\infty}$ of models of (X, D) with the following properties:

- (1) For each $n \geq 1$, \mathcal{D}_n is relatively nef with respect to $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$ (cf Conventions and terminology 0.5.6).
- (2) If we set $\phi_n = g(\mathcal{X}_n, \mathcal{D}_n) - g$, then $\lim_{n \rightarrow \infty} \|\phi_n\|_{\text{sup}} = 0$.

As an application of results in Appendix (cf. Corollary A.3.2), we have the following characterization of relatively nef divisors.

Proposition 2.1.7. *Let \mathcal{X} be a normal model of X and let \mathcal{D} be an \mathbb{R} -Cartier divisor on \mathcal{X} . Then $g_{(\mathcal{X}, \mathcal{D})}$ is of $(C^0 \cap \text{PSH})$ -type if and only if \mathcal{D} is relatively nef.*

In addition, we have the following propositions.

Proposition 2.1.8. *We assume that v is non-trivial. Let D be a nef \mathbb{R} -Cartier divisor on X . Let g be a D -Green function of $(C^0 \cap \text{PSH})$ -type. Then there are sequences $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^\infty$ and $\{(\mathcal{X}_n, \mathcal{D}'_n)\}_{n=1}^\infty$ of models of (X, D) with the following properties:*

- (1) *For all $n \geq 1$, \mathcal{D}_n and \mathcal{D}'_n are relatively nef with respect to $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$.*
- (2) *$g_{(\mathcal{X}_n, \mathcal{D}_n)} \leq g \leq g_{(\mathcal{X}_n, \mathcal{D}'_n)}$ for all $n \geq 1$.*
- (3) *If we set $\phi_n = g_{(\mathcal{X}_n, \mathcal{D}_n)} - g$ and $\phi'_n = g_{(\mathcal{X}_n, \mathcal{D}'_n)} - g$, then*

$$\lim_{n \rightarrow \infty} \|\phi_n\|_{\sup} = \lim_{n \rightarrow \infty} \|\phi'_n\|_{\sup} = 0.$$

Proof. By its definition, there is a sequence $\{(\mathcal{X}_n, \mathcal{D}''_n)\}_{n=1}^\infty$ of models of (X, D) with the following properties:

- (i) *For all $n \geq 1$, \mathcal{D}''_n is relatively nef with respect to $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$.*
- (ii) *If we set $\phi''_n = g_{(\mathcal{X}_n, \mathcal{D}''_n)} - g$, then $\lim_{n \rightarrow \infty} \|\phi''_n\|_{\sup} = 0$.*

Here we set

$$\mathcal{D}_n := \mathcal{D}''_n - \frac{\|\phi''_n\|_{\sup}}{-2 \log v(\varpi)} (\mathcal{X}_n)_\circ \quad \text{and} \quad \mathcal{D}'_n := \mathcal{D}''_n + \frac{\|\phi''_n\|_{\sup}}{-2 \log v(\varpi)} (\mathcal{X}_n)_\circ,$$

where $(\mathcal{X}_n)_\circ$ is the central fiber of $\mathcal{X}_n \rightarrow \text{Spec}(k^\circ)$. Then

$$g_{(\mathcal{X}_n, \mathcal{D}_n)} = g_{(\mathcal{X}_n, \mathcal{D}''_n)} - \|\phi''_n\|_{\sup} \quad \text{and} \quad g_{(\mathcal{X}_n, \mathcal{D}'_n)} = g_{(\mathcal{X}_n, \mathcal{D}''_n)} + \|\phi''_n\|_{\sup},$$

and hence

$$g_{(\mathcal{X}_n, \mathcal{D}_n)} - g = \phi''_n - \|\phi''_n\|_{\sup} \leq 0 \quad \text{and} \quad g_{(\mathcal{X}_n, \mathcal{D}'_n)} - g = \phi''_n + \|\phi''_n\|_{\sup} \geq 0,$$

as required. \square

2.2. Definition of adelic \mathbb{R} -Cartier divisors. We assume that k° is excellent. Let X be a projective, smooth and geometrically integral variety over k . Let k_ν be the completion of k with respect to ν . By abuse of notation, the unique extension of ν to k_ν is also denoted by ν . We set

$$X_\nu := X \times_{\text{Spec}(k)} \text{Spec}(k_\nu),$$

which is also a projective, smooth and geometrically integral variety over k_ν .

A pair $\overline{D} = (D, g)$ is called an *adelic \mathbb{R} -Cartier divisor of C^0 -type on X* if D is an \mathbb{R} -Cartier divisor on X and g is a D -Green function of C^0 -type on X_ν^{an} . If D is nef and g is of $(\text{PSH} \cap C^0)$ -type, then \overline{D} is said to be *relatively nef*. Moreover, we say \overline{D} is *integrable* if there are relatively nef adelic \mathbb{R} -Cartier divisors \overline{D}' and \overline{D}'' of C^0 -type on X such that $\overline{D} = \overline{D}' - \overline{D}''$. We say a continuous function ϕ on X_ν^{an} is *integrable* if $(0, \phi)$ is integrable as an adelic \mathbb{R} -Cartier divisor. Let $\overline{D}' = (D', g')$ be another adelic \mathbb{R} -Cartier divisor of C^0 -type on X . For $a, a' \in \mathbb{R}$, we define $a\overline{D} + a'\overline{D}'$ to be

$$a\overline{D} + a'\overline{D}' := (aD + a'D', ag + a'g').$$

The space of all adelic \mathbb{R} -Cartier divisors of C^0 -type is denoted by $\text{Div}_{C^0}^a(X)_{\mathbb{R}}$, which forms a vector space over \mathbb{R} by the above formula. For $\overline{D}_1 = (D_1, g_1), \overline{D}_2 = (D_2, g_2) \in \text{Div}_{C^0}^a(X)_{\mathbb{R}}$, we define $\overline{D}_1 \leq \overline{D}_2$ to be

$$\overline{D}_1 \leq \overline{D}_2 \stackrel{\text{def}}{\iff} D_1 \leq D_2 \text{ and } g_1 \leq g_2.$$

Let \mathcal{X} be a normal model of X over $\text{Spec}(k^\circ)$ and let \mathcal{D} be an \mathbb{R} -Cartier divisor on \mathcal{X} . The pair $(\mathcal{X}, \mathcal{D})$ gives rise to an adelic \mathbb{R} -Cartier divisor of C^0 -type on X , that is, the pair $(\mathcal{D} \cap X, g_{(\mathcal{X}, \mathcal{D})})$ of $\mathcal{D} \cap X$ and $g_{(\mathcal{X}, \mathcal{D})}$. We denote it by \mathcal{D}^a and it is called the *associated adelic \mathbb{R} -Cartier divisor with \mathcal{D}* . By abuse of notation, we often use the notations $\mathcal{D} \leq \overline{D}_2$ and $\overline{D}_1 \leq \mathcal{D}$ instead of $\mathcal{D}^a \leq \overline{D}_2$ and $\overline{D}_1 \leq \mathcal{D}^a$, respectively

Proposition 2.2.1. *Let \mathcal{X} be a normal model of X over $\text{Spec}(k^\circ)$ and let $\text{Div}(\mathcal{X})_{\mathbb{R}}$ be the group of \mathbb{R} -Cartier divisors on \mathcal{X} . Let $\iota : \text{Div}(\mathcal{X})_{\mathbb{R}} \rightarrow \text{Div}_{C^0}^a(X)_{\mathbb{R}}$ be the map given by $\mathcal{D} \mapsto \mathcal{D}^a$. Then we have the following:*

- (1) *The map $\iota : \text{Div}(\mathcal{X})_{\mathbb{R}} \rightarrow \text{Div}_{C^0}^a(X)_{\mathbb{R}}$ is an injective homomorphism of \mathbb{R} -vector spaces.*
- (2) $\mathcal{D}_1 \leq \mathcal{D}_2 \iff \mathcal{D}_1^a \leq \mathcal{D}_2^a$.

Proof. Clearly ι is a homomorphism of \mathbb{R} -vector spaces. (2) is a consequence of Proposition 2.1.4. The injectivity of ι follows from (2). \square

2.3. Local degree. We assume that k° is excellent and k is perfect. We use the same notation as in Subsection 2.2. Let x be a closed point of X with $x \notin \text{Supp}_{\mathbb{R}}(D)$. Let $k(x)$ be the residue field at x . As $k(x)$ is separable over k , we have $k(x) \otimes_k k_\nu = k_1 \oplus \cdots \oplus k_l$ for some finite separable extensions k_1, \dots, k_l over k_ν . Note that each k_i has the unique extension v_i of v . The *local degree of \overline{D} along x over v* is defined by

$$\widehat{\deg}_v(\overline{D}|_x) := \sum_{i=1}^l \frac{[k_i : k_\nu]}{2} g(v_i).$$

Here we assume that k is a number field and $v(f) = \#(O_k/P)^{-\text{ord}_P(f)}$, where O_k is the ring of integers in k and P is a maximal ideal of O_k . Let \mathcal{X} be a normal model of X over $\text{Spec}((O_k)_P)$. Let $\mathcal{D} = a_1 \mathcal{D}_1 + \cdots + a_l \mathcal{D}_r$ be an \mathbb{R} -Cartier divisor on \mathcal{X} such that $\mathcal{D} \cap X = D$, $a_1, \dots, a_r \in \mathbb{R}$ and $\mathcal{D}_1, \dots, \mathcal{D}_r$ are effective Cartier divisors on \mathcal{X} . We assume that $g = g_{(\mathcal{X}, \mathcal{D})}$ and $x \notin \text{Supp}_{\mathbb{Z}}(\mathcal{D}_1) \cup \cdots \cup \text{Supp}_{\mathbb{Z}}(\mathcal{D}_r)$. Let $O_{k(x)}$ be the ring of integers in $k(x)$. Then it is easy to see that

$$(2.3.1) \quad \widehat{\deg}_v(\overline{D}|_x) = \sum_{j=1}^r a_j \log \# \left((O_{k(x)}(\mathcal{D}_j) / O_{k(x)})_P \right).$$

2.4. Local intersection number. We assume that k° is excellent and v is non-trivial. We use the same notation as in Subsection 2.2. Let ϕ be a continuous function on X_v^{an} . Let \mathcal{X} be a normal model of X and let $\mathcal{L}_1, \dots, \mathcal{L}_d$ be \mathbb{R} -Cartier divisors on \mathcal{X} . Let $\Gamma_1, \dots, \Gamma_r$ be irreducible components of the central fiber \mathcal{X}_\circ of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$. Let v_j be the discrete valuation arising from Γ_j such that $v_j|_k = v$. By Lemma 1.3.7, there is a unique $\tilde{v}_j \in X_v^{\text{an}}$ such that the restriction of \tilde{v}_j to $\text{Rat}(X)$ is v_j . For each i and j , we can choose a unique real number λ_{ij} such that $\Gamma_j \not\subseteq \text{Supp}_W(\mathcal{L}_i + \lambda_{ij} \mathcal{X}_\circ)$ (for Supp_W , see Definition 1.1.2). Then the number given by

$$\sum_{j=1}^r \frac{\phi(\tilde{v}_j) \text{ord}_{\Gamma_j}(\varpi)}{-2 \log v(\varpi)} \deg_{k^\circ/k^\circ} \left((\mathcal{L}_1 + \lambda_{1j} \mathcal{X}_\circ)|_{\Gamma_j} \cdots (\mathcal{L}_d + \lambda_{dj} \mathcal{X}_\circ)|_{\Gamma_j} \right)$$

is denoted by $\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi)$, where \deg_{k°/k° is the degree over k°/k° . Obviously, $\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi)$ is multi-linear with respect to $\mathcal{L}_1, \dots, \mathcal{L}_d$. In addition,

$$\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; a\phi + a'\phi') = a \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi) + a' \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi')$$

for $a, a' \in \mathbb{R}$ and $\phi, \phi' \in C^0(X_v^{\text{an}})$. Let \mathcal{E} be a vertical \mathbb{R} -Cartier divisor on \mathcal{X} and let $\mathcal{E} = a_1 \Gamma_1 + \cdots + a_r \Gamma_r$ be the irreducible decomposition of \mathcal{E} as a Weil divisor. Let $\phi_{\mathcal{E}}$ be the continuous function arising from \mathcal{E} , that is, $\phi_{\mathcal{E}} = g_{(\mathcal{X}, \mathcal{E})}$. Then

$$\begin{aligned} \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi_{\mathcal{E}}) &= \sum_{j=1}^r a_j \deg_{k^\circ/k^\infty} \left((\mathcal{L}_1 + \lambda_{1j} \mathcal{X}_\circ) \big|_{\Gamma_j} \cdots (\mathcal{L}_d + \lambda_{dj} \mathcal{X}_\circ) \big|_{\Gamma_j} \right) \\ &= \deg_{k^\circ/k^\infty}(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot \mathcal{E}). \end{aligned}$$

If $\mathcal{L}_1, \dots, \mathcal{L}_d$ are relatively nef and $\phi \leq \phi'$, then

$$(2.4.1) \quad \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi) \leq \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi').$$

Let us begin with the following lemma:

Lemma 2.4.2. *Let $\mathcal{L}_1, \dots, \mathcal{L}_{d+2}, \mathcal{L}'_1, \dots, \mathcal{L}'_{d+2}$ be relatively nef \mathbb{R} -Cartier divisors on \mathcal{X} . We assume that there are $a_1, \dots, a_d, a_{d+1} \in \mathbb{R}_{\geq 0}$ with the following properties:*

- (1) *For all $i = 1, \dots, d$, $\mathcal{L}_i \cap X = \mathcal{L}'_i \cap X$ and $-a_i \mathcal{X}_\circ \leq \mathcal{L}'_i - \mathcal{L}_i \leq a_i \mathcal{X}_\circ$.*
- (2) *$\mathcal{L}_{d+1} \cap X = \mathcal{L}_{d+2} \cap X$ and $\mathcal{L}'_{d+1} \cap X = \mathcal{L}'_{d+2} \cap X$. Moreover,*

$$-2a_{d+1} g_{(\mathcal{X}, \mathcal{X}_\circ)} \leq \psi' - \psi \leq 2a_{d+1} g_{(\mathcal{X}, \mathcal{X}_\circ)},$$

$$\text{where } \psi := g_{(\mathcal{X}, \mathcal{L}_{d+1} - \mathcal{L}_{d+2})} \text{ and } \psi' := g_{(\mathcal{X}, \mathcal{L}'_{d+1} - \mathcal{L}'_{d+2})}.$$

Then we have the following inequality:

$$\left| \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi) - \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi') \right| \leq \sum_{i=1}^{d+1} 2a_i \deg(L_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_{d+1}),$$

where $L_i := \mathcal{L}_i \cap X$ for $i = 1, \dots, d+1$.

Proof. As $\mathcal{L}'_1, \dots, \mathcal{L}'_d$ are relatively nef and $-2a_{d+1} g_{(\mathcal{X}, \mathcal{X}_\circ)} \leq \psi' - \psi \leq 2a_{d+1} g_{(\mathcal{X}, \mathcal{X}_\circ)}$, by using (2.4.1), we have

$$\left| \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi') - \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi) \right| \leq 2a_{d+1} \deg(L_1 \cdots L_d),$$

and hence,

$$\begin{aligned} \left| \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi) - \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi') \right| &\leq \left| \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi) - \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi) \right| + 2a_{d+1} \deg(L_1 \cdots L_d). \end{aligned}$$

On the other hand, by Lemma 1.3.4,

$$\begin{aligned} \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_d; \psi) &= \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi) \\ &\quad + \sum_{i=1}^d \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{E}_i \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \psi), \end{aligned}$$

where $\mathcal{E}_i = \mathcal{L}'_i - \mathcal{L}_i$. Let ϕ_i be the continuous function arising from \mathcal{E}_i . Then, as

$$\begin{aligned} \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{E}_i \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \psi) &= \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot (\mathcal{L}_{d+1} - \mathcal{L}_{d+2}) \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \phi_i) \end{aligned}$$

and $-a_i g_{(\mathcal{X}, \mathcal{X}_\circ)} \leq \phi_i \leq a_i g_{(\mathcal{X}, \mathcal{X}_\circ)}$, by using (2.4.1), we can see that

$$\begin{aligned} & \left| \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{E}_i \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \psi) \right| \\ & \leq \left| \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{L}_{d+1} \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \phi_i) \right| \\ & \quad + \left| \widehat{\deg}_v(\mathcal{L}'_1 \cdots \mathcal{L}'_{i-1} \cdot \mathcal{L}_{d+2} \cdot \mathcal{L}_{i+1} \cdots \mathcal{L}_d; \phi_i) \right| \\ & \leq 2a_i \deg(L_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_d \cdot L_{d+1}), \end{aligned}$$

as desired. \square

The next proposition guarantees the intersection pairing of integrable adelic \mathbb{R} -Cartier divisors along an integrable continuous function.

Proposition-Definition 2.4.3. *Let $\bar{L}_1 = (L_1, g_1), \dots, \bar{L}_d = (L_d, g_d)$ be relatively nef adelic \mathbb{R} -Cartier divisors on X , and let ϕ be an integrable continuous function on X_v^{an} . Then there are sequences*

$$\begin{aligned} & \{(\mathcal{X}_{1,n}, \mathcal{L}_{1,n})\}_{n=1}^\infty, \dots, \{(\mathcal{X}_{d,n}, \mathcal{L}_{d,n})\}_{n=1}^\infty, \\ & \{(\mathcal{X}_{d+1,n}, \mathcal{L}_{d+1,n})\}_{n=1}^\infty, \{(\mathcal{X}_{d+2,n}, \mathcal{L}_{d+2,n})\}_{n=1}^\infty \end{aligned}$$

with the following properties:

- (1) $\mathcal{X}_{i,n}$ is a normal model of X over $\text{Spec}(k^\circ)$ and $\mathcal{L}_{i,n}$ is a relatively nef \mathbb{R} -Cartier divisor on $\mathcal{X}_{i,n}$ for $i = 1, \dots, d+2$ and $n \geq 1$.
- (2) $\mathcal{L}_{i,n} \cap X = L_i$ for $i = 1, \dots, d$ and $n \geq 1$.
- (3) There is an \mathbb{R} -Cartier divisor L_{d+1} on X such that $L_{d+1} = \mathcal{L}_{d+1,n} \cap X = \mathcal{L}_{d+2,n} \cap X$ for all $n \geq 1$.
- (4) If we set $\phi_{i,n} := g_i - g_{(\mathcal{X}_{i,n}, \mathcal{L}_{i,n})}$, then $\lim_{n \rightarrow \infty} \|\phi_{i,n}\|_{\text{sup}} = 0$ for $i = 1, \dots, d$.
- (5) If we set $\psi_n := g_{(\mathcal{X}_{d+1,n}, \mathcal{L}_{d+1,n})} - g_{(\mathcal{X}_{d+2,n}, \mathcal{L}_{d+2,n})}$, then

$$\lim_{n \rightarrow \infty} \|\psi_n - \phi\|_{\text{sup}} = 0.$$

For sequences

$$\begin{aligned} & \{(\mathcal{X}_{1,n}, \mathcal{L}_{1,n})\}_{n=1}^\infty, \dots, \{(\mathcal{X}_{d,n}, \mathcal{L}_{d,n})\}_{n=1}^\infty, \\ & \{(\mathcal{X}_{d+1,n}, \mathcal{L}_{d+1,n})\}_{n=1}^\infty, \{(\mathcal{X}_{d+2,n}, \mathcal{L}_{d+2,n})\}_{n=1}^\infty \end{aligned}$$

satisfying the above properties, let \mathcal{Y}_n be a normal model of X over $\text{Spec}(k^\circ)$ together with birational morphisms

$$\mu_{i,n} : \mathcal{Y}_n \rightarrow \mathcal{X}_{i,n}$$

for $i = 1, \dots, d$. Then the following limits

$$\begin{cases} \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \phi), \\ \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \psi_n) \end{cases}$$

exist and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \phi) \\ & = \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \psi_n). \end{aligned}$$

Moreover, the above limits do not depend on the choice of the sequences

$$\{(\mathcal{X}_{1,n}, \mathcal{L}_{1,n})\}_{n=1}^{\infty}, \dots, \{(\mathcal{X}_{d,n}, \mathcal{L}_{d,n})\}_{n=1}^{\infty}, \\ \{(\mathcal{X}_{d+1,n}, \mathcal{L}_{d+1,n})\}_{n=1}^{\infty}, \{(\mathcal{X}_{d+2,n}, \mathcal{L}_{d+2,n})\}_{n=1}^{\infty},$$

so that this limit is denoted by $\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi)$.

Proof. The existence of sequences are obvious by the relative nefness of $\bar{L}_1, \dots, \bar{L}_d$ and the integrability of ϕ .

We set

$$\begin{cases} A_n = \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \phi), \\ B_n = \widehat{\deg}_v(\mu_{1,n}^*(\mathcal{L}_{1,n}) \cdots \mu_{d,n}^*(\mathcal{L}_{d,n}); \psi_n). \end{cases}$$

For a positive number ϵ , there is N such that

$$\|\psi_n - \phi\|_{\sup} \leq 2\epsilon(-\log v(\varpi)) \quad \text{and} \quad \|\phi_{i,n}\|_{\sup} \leq \epsilon(-\log v(\varpi))$$

for all $n \geq N$ and $i = 1, \dots, d$. Then, for $n, m \geq N$,

$$\begin{aligned} \left| g(\mathcal{X}_{i,n}, \mathcal{L}_{i,n}) - g(\mathcal{X}_{i,m}, \mathcal{L}_{i,m}) \right| &\leq \left| g_i - g(\mathcal{X}_{i,n}, \mathcal{L}_{i,n}) \right| \\ &\quad + \left| g_i - g(\mathcal{X}_{i,m}, \mathcal{L}_{i,m}) \right| \leq 2\epsilon(-\log v(\varpi)) \end{aligned}$$

for $i = 1, \dots, d$. Let us choose a normal model $\mathcal{X}_{n,m}$ of X together with birational morphisms $\tau_n : \mathcal{X}_{n,m} \rightarrow \mathcal{Y}_n$ and $\tau_m : \mathcal{X}_{n,m} \rightarrow \mathcal{Y}_m$. Then the above inequality implies

$$-\epsilon g(\mathcal{X}_{n,m}, (\mathcal{X}_{n,m})_{\circ}) \leq g(\mathcal{X}_{n,m}, \tau_n^*(\mu_{i,n}^*(\mathcal{L}_{i,n})) - \tau_m^*(\mu_{i,m}^*(\mathcal{L}_{i,m}))) \leq \epsilon g(\mathcal{X}_{n,m}, (\mathcal{X}_{n,m})_{\circ}),$$

so that, by Proposition 2.2.1,

$$-\epsilon(\mathcal{X}_{n,m})_{\circ} \leq \tau_n^*(\mu_{i,n}^*(\mathcal{L}_{i,n})) - \tau_m^*(\mu_{i,m}^*(\mathcal{L}_{i,m})) \leq \epsilon(\mathcal{X}_{n,m})_{\circ}$$

for $i = 1, \dots, d$. On the other hand,

$$\|\psi_n - \psi_m\|_{\sup} \leq \|\psi_n - \phi\|_{\sup} + \|\psi_m - \phi\|_{\sup} \leq 4\epsilon(-\log v(\varpi)).$$

Therefore, by using Lemma 2.4.2, we have

$$|B_n - B_m| \leq 2\epsilon \sum_{i=1}^{d+1} \deg(L_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_{d+1})$$

for $n, m \geq N$, which shows that the sequence $\{B_n\}_{n=1}^{\infty}$ is a Cauchy sequence, so that its limit exists. Further, as $0 \leq |\phi - \psi_n| \leq 2\epsilon(-\log v(\varpi))$, by (2.4.1), we have

$$|A_n - B_n| \leq \epsilon \deg(L_1 \cdots L_d),$$

so that $\lim_{n \rightarrow \infty} A_n$ exists and $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} B_n$.

Let $\{(\mathcal{X}'_{1,n}, \mathcal{L}'_{1,n})\}_{n=1}^{\infty}, \dots, \{(\mathcal{X}'_{d+2,n}, \mathcal{L}'_{d+2,n})\}_{n=1}^{\infty}$ be another sequences satisfying the above properties (1), (2), (3), (4) and (5). For the above sequences, L_{d+1} in the property (3), $\phi_{i,n}$ in the property (4) and ψ_n in the property (5) are denoted by L'_{d+1} , $\phi'_{i,n}$ and ψ'_n , respectively. Replacing \mathcal{Y}_n by a suitable model of X , we may assume that there

are birational morphisms $\mu'_{i,n} : \mathcal{Y}_n \rightarrow \mathcal{X}'_{i,n}$ ($i = 1, \dots, d$). For a positive number ϵ , there is N such that

$$\begin{cases} \|\psi_n - \phi\|_{\sup} \leq 2\epsilon(-\log v(\varpi)), & \|\phi_{i,n}\|_{\sup} \leq \epsilon(-\log v(\varpi)), \\ \|\psi'_n - \phi\|_{\sup} \leq 2\epsilon(-\log v(\varpi)), & \|\phi'_{i,n}\|_{\sup} \leq \epsilon(-\log v(\varpi)) \end{cases}$$

for all $n \geq N$ and $i = 1, \dots, d$. Then,

$$\begin{aligned} & \left| \mathcal{G}_{(\mathcal{Y}_n, \mu_{i,n}^*(\mathcal{L}_{i,n}))} - \mathcal{G}_{(\mathcal{Y}_n, \mu_{i,n}^*(\mathcal{L}'_{i,n}))} \right| \\ & \leq \left| g_i - \mathcal{G}_{(\mathcal{Y}_n, \mu_{i,n}^*(\mathcal{L}_{i,n}))} \right| + \left| g_i - \mathcal{G}_{(\mathcal{Y}_n, \mu_{i,n}^*(\mathcal{L}'_{i,n}))} \right| \leq 2\epsilon(-\log v(\varpi)) \end{aligned}$$

for $n \geq N$ and $i = 1, \dots, d$. Therefore, in the similar way as before,

$$-\epsilon(\mathcal{Y}_n)_\circ \leq \mu_{i,n}^*(\mathcal{L}_{i,n}) - \mu_{i,n}^*(\mathcal{L}'_{i,n}) \leq \epsilon(\mathcal{Y}_n)_\circ.$$

Moreover,

$$\|\psi_n - \psi'_n\|_{\sup} \leq \|\psi_n - \phi\|_{\sup} + \|\psi'_n - \phi\|_{\sup} \leq 4\epsilon(-\log v(\varpi))$$

for $n \geq N$, and hence the uniqueness of the limit follows from Lemma 2.4.2. \square

Let ϕ be an integrable continuous function on X_v^{an} . Let $\bar{L}_1, \dots, \bar{L}_i, \bar{L}'_i, \dots, \bar{L}_d$ be relatively nef adelic \mathbb{R} -Cartier divisors of C^0 -type on X . Then it is easy to see that

$$\begin{aligned} (2.4.4) \quad \widehat{\deg}_v(\bar{L}_1 \cdots (a\bar{L}_i + a'\bar{L}'_i) \cdots \bar{L}_d; \phi) \\ = a\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi) + a'\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}'_i \cdots \bar{L}_d; \phi) \end{aligned}$$

for $a, a' \in \mathbb{R}_{\geq 0}$, and that

$$(2.4.5) \quad \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_j \cdots \bar{L}_d; \phi) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_j \cdots \bar{L}_i \cdots \bar{L}_d; \phi).$$

Let $\bar{L}_1, \dots, \bar{L}_d$ be integrable adelic \mathbb{R} -Cartier divisors of C^0 -type on X . For each i , we choose relatively nef adelic \mathbb{R} -Cartier divisors \bar{L}_{i+1} and \bar{L}_{i-1} of C^0 -type such that $\bar{L}_i = \bar{L}_{i+1} - \bar{L}_{i-1}$. By using (2.4.4), it is not difficult to see that the quantity

$$\sum_{\epsilon_1, \dots, \epsilon_{d+1} \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_{d+1} \widehat{\deg}_v(\bar{L}_{1, \epsilon_1} \cdots \bar{L}_{d, \epsilon_d}; \phi)$$

does not depend on the choice of $\bar{L}_{1,+1}, \bar{L}_{1,-1}, \dots, \bar{L}_{d,+1}, \bar{L}_{d,-1}$, so that it is denoted by $\widehat{\deg}(\bar{L}_1 \cdots \bar{L}_d; \phi)$. Further, (2.4.4) and (2.4.5) extends to the following formula:

$$\begin{aligned} (2.4.6) \quad \widehat{\deg}_v(\bar{L}_1 \cdots (a\bar{L}_i + a'\bar{L}'_i) \cdots \bar{L}_d; \phi) \\ = a\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi) + a'\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}'_i \cdots \bar{L}_d; \phi) \end{aligned}$$

and

$$(2.4.7) \quad \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_j \cdots \bar{L}_d; \phi) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_j \cdots \bar{L}_i \cdots \bar{L}_d; \phi)$$

for $a, a' \in \mathbb{R}$ and integrable adelic \mathbb{R} -Cartier divisors $\bar{L}_1, \dots, \bar{L}_i, \bar{L}'_i, \dots, \bar{L}_d$ of C^0 -type on X . Here let us consider a consequence of Proposition 2.4.3.

Proposition 2.4.8. *Let $\bar{L}_1 = (L_1, g_1), \dots, \bar{L}_d = (L_d, g_d)$ be integrable adelic \mathbb{R} -Cartier divisors of C^0 -type on X , and let ϕ be an integrable continuous function on X_v^{an} . If $L_i = 0$, then*

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi) = \widehat{\deg}_v(\bar{L}_1 \cdots (0, \phi) \cdots \bar{L}_d; g_i).$$

Proof. For $j = 1, \dots, d$, let

$$\bar{L}_{j,+1} = (L_{j,+1}, g_{j,+1}) \quad \text{and} \quad \bar{L}_{j,-1} = (L_{j,-1}, g_{j,-1})$$

be relatively nef adelic \mathbb{R} -Cartier divisors of C^0 -type on X such that $\bar{L}_j = \bar{L}_{j,+1} - \bar{L}_{j,-1}$. Moreover, we choose relative nef adelic \mathbb{R} -Cartier divisors

$$\bar{L}_{d+1,+1} = (L_{d+1,+1}, g_{d+1,+1}) \quad \text{and} \quad \bar{L}_{d+1,-1} = (L_{d+1,-1}, g_{d+1,-1})$$

of C^0 -type on X such that $\bar{L}_{d+1,+1} - \bar{L}_{d+1,-1} = (0, \phi)$. Then there are sequences

$$\{(\mathcal{X}_{1,+1,n}, \mathcal{L}_{1,+1,n})\}_{n=1}^{\infty}, \{(\mathcal{X}_{1,-1,n}, \mathcal{L}_{1,-1,n})\}_{n=1}^{\infty}, \dots, \\ \{(\mathcal{X}_{d+1,+1,n}, \mathcal{L}_{d+1,+1,n})\}_{n=1}^{\infty}, \{(\mathcal{X}_{d+1,-1,n}, \mathcal{L}_{d+1,-1,n})\}_{n=1}^{\infty}$$

satisfying the following conditions:

- (a) $\mathcal{X}_{j,\epsilon,n}$ is a normal model of X over $\text{Spec}(k^\circ)$ for $j = 1, \dots, d+1$, $\epsilon = \pm 1$ and $n \geq 1$.
- (b) $\mathcal{L}_{j,\epsilon,n}$ is a nef \mathbb{R} -Cartier divisor on $\mathcal{X}_{j,\epsilon,n}$ such that $\mathcal{L}_{j,\epsilon,n} \cap X = L_{j,\epsilon}$ for $j = 1, \dots, d+1$, $\epsilon = \pm 1$ and $n \geq 1$.
- (c) If we set $\phi_{j,\epsilon,n} = g_{j,\epsilon} - g(\mathcal{X}_{j,\epsilon,n}, \mathcal{L}_{j,\epsilon,n})$, then $\lim_{n \rightarrow \infty} \|\phi_{j,\epsilon,n}\|_{\text{sup}} = 0$ for $j = 1, \dots, d+1$ and $\epsilon = \pm 1$.

Here we set

$$\begin{cases} \psi_n = g(\mathcal{X}_{d+1,+1,n}, \mathcal{L}_{d+1,+1,n}) - g(\mathcal{X}_{d+1,-1,n}, \mathcal{L}_{d+1,-1,n}), \\ \theta_n = g(\mathcal{X}_{i,+1,n}, \mathcal{L}_{i,+1,n}) - g(\mathcal{X}_{i,-1,n}, \mathcal{L}_{i,-1,n}). \end{cases}$$

Then, by Proposition-Definition 2.4.3,

$$\lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{L}_{1,\epsilon_1,n} \cdots \mathcal{L}_{i,\epsilon_i,n} \cdots \mathcal{L}_{d,\epsilon_d,n}; \psi_n) = \widehat{\deg}_v(\bar{L}_{1,\epsilon_1} \cdots \bar{L}_{i,\epsilon_i} \cdots \bar{L}_{d,\epsilon_d}; \phi)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{L}_{1,\epsilon_1,n} \cdots \mathcal{L}_{d+1,\epsilon_{d+1},n} \cdots \mathcal{L}_{d,\epsilon_d,n}; \theta_n) \\ = \widehat{\deg}_v(\bar{L}_{1,\epsilon_1} \cdots \bar{L}_{d+1,\epsilon_{d+1}} \cdots \bar{L}_{d,\epsilon_d}; g_i). \end{aligned}$$

Therefore, if we set $\mathcal{L}_{i,n} = \mathcal{L}_{i,+1,n} - \mathcal{L}_{i,-1,n}$ for $i = 1, \dots, d+1$, then

$$\begin{cases} \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{L}_{1,n} \cdots \mathcal{L}_{i,n} \cdots \mathcal{L}_{d,n}; \psi_n) = \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_i \cdots \bar{L}_d; \phi), \\ \lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{L}_{1,n} \cdots \mathcal{L}_{d+1,n} \cdots \mathcal{L}_{d,n}; \theta_n) = \widehat{\deg}_v(\bar{L}_1 \cdots (0, \phi) \cdots \bar{L}_d; g_i). \end{cases}$$

On the other hand, note that

$$\widehat{\deg}_v(\mathcal{L}_{1,n} \cdots \mathcal{L}_{i,n} \cdots \mathcal{L}_{d,n}; \psi_n) = \widehat{\deg}_v(\mathcal{L}_{1,n} \cdots \mathcal{L}_{d+1,n} \cdots \mathcal{L}_{d,n}; \theta_n).$$

Thus the assertion of the proposition follows. \square

The results of this subsection leads to the following definition:

Definition 2.4.9. Let $\bar{L}_1 = (L_1, g_1), \dots, \bar{L}_{d+1} = (L_{d+1}, g_{d+1})$ be integrable adelic \mathbb{R} -Cartier divisors of C^0 -type on X . By Proposition 2.4.8 and (2.4.7), if $L_i = 0$ for some i , then

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_{d+1})$$

is well-defined, that is,

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_{d+1}) := \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_{i-1} \cdot \bar{L}_{i+1} \cdots \bar{L}_{d+1}; g_i).$$

Moreover, it is symmetric and multi-linear.

Proposition 2.4.10. *Let $\bar{L}_1, \dots, \bar{L}_d$ be integrable adelic \mathbb{R} -Cartier divisors of C^0 -type on X , and let ϕ and ϕ' be continuous functions on X_v^{an} . Then we have the following:*

- (1) *For $f \in \text{Rat}(X)_{\mathbb{R}}^\times$, $\widehat{\deg}_v(((f), -\log|f|^2) \cdot \bar{L}_1 \cdots \bar{L}_d; \phi) = 0$.*
- (2) *If $\bar{L}_1, \dots, \bar{L}_d$ are relatively nef and $\phi' \leq \phi$ on X_v^{an} , then*

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi') \leq \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi).$$

- (3) *Let*

$$\begin{aligned} \bar{L}_{1,1} &= (L_{1,1}, g_{1,1}), \bar{L}_{1,-1} = (L_{1,-1}, g_{1,-1}), \dots, \\ \bar{L}_{d,1} &= (L_{d,1}, g_{d,1}), \bar{L}_{d,-1} = (L_{d,-1}, g_{d,-1}) \end{aligned}$$

be relatively nef adelic \mathbb{R} -Cartier divisors of C^0 -type on X such that $\bar{L}_i = \bar{L}_{i,1} - \bar{L}_{i,-1}$ for $i = 1, \dots, d$. Then

$$\left| \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) \right| \leq \frac{\|\phi\|_{\sup}}{-2 \log v(\varpi)} \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \deg(L_{1,\epsilon_1} \cdots L_{d,\epsilon_d}).$$

Proof. Let \mathcal{X} be a normal model of X over $\text{Spec}(k^\circ)$ and let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_d$ be \mathbb{R} -Cartier divisors on \mathcal{X}

(1) By the definition of $\widehat{\deg}_v$, we have $\widehat{\deg}_v(((f)_{\mathcal{X}} \cdot \mathcal{L}_1 \cdots \mathcal{L}_d; \phi) = 0$. Thus (1) follows.

(2) If $\mathcal{L}_1, \dots, \mathcal{L}_d$ are relatively nef, then $\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi') \leq \widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi)$ (cf. (2.4.1)), so that we have (2).

(3) First of all, note that

$$\widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) = \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_d \widehat{\deg}_v(\bar{L}_{1,\epsilon_1} \cdots \bar{L}_{d,\epsilon_d}; \phi).$$

Therefore, by using (2),

$$\begin{aligned} \left| \widehat{\deg}_v(\bar{L}_1 \cdots \bar{L}_d; \phi) \right| &\leq \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \left| \widehat{\deg}_v(\bar{L}_{1,\epsilon_1} \cdots \bar{L}_{d,\epsilon_d}; \phi) \right| \\ &\leq \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \widehat{\deg}_v(\bar{L}_{1,\epsilon_1} \cdots \bar{L}_{d,\epsilon_d}; \|\phi\|_{\sup}) \\ &= \frac{\|\phi\|_{\sup}}{-2 \log v(\varpi)} \sum_{\epsilon_1, \dots, \epsilon_d \in \{\pm 1\}} \deg(L_{1,\epsilon_1} \cdots L_{d,\epsilon_d}), \end{aligned}$$

as required. □

Finally let us consider Zariski's lemma for integrable functions. We assume that $\dim X = 1$ and v is complete. Let $C_{\text{int}}^0(X^{\text{an}})$ be the space of integrable continuous functions on X^{an} . Let

$$\langle \cdot, \cdot \rangle : C_{\text{int}}^0(X^{\text{an}}) \times C_{\text{int}}^0(X^{\text{an}}) \rightarrow \mathbb{R}$$

be a map given by $\langle \varphi, \psi \rangle := \widehat{\deg}_v((0, \varphi); \psi) = \widehat{\deg}_v((0, \varphi) \cdot (0, \psi))$, which is bilinear and symmetric (see Definition 2.4.9).

Lemma 2.4.11 (Zariski's lemma for integrable functions). *The above pairing $\langle \cdot, \cdot \rangle$ is negative semi-definite. Moreover, for $\varphi \in C_{\text{int}}^0(X^{\text{an}})$, $\langle \varphi, \varphi \rangle = 0$ if and only if φ is a constant function.*

Proof. The proof can be found in [32]. For reader's convenience, we prove it here. Let $\mathbb{M}(X^{\text{an}})$ be the group of model functions on X^{an} (cf. Subsection 3.2). Then, by virtue of Zariski's lemma (cf. [27, Lemma 1.1.4]), for $\theta \in \mathbb{M}(X^{\text{an}}) \otimes_{\mathbb{Z}} \mathbb{R}$, $\langle \theta, \theta \rangle \leq 0$. Therefore, by the density theorem (cf. Theorem 3.3.3) together with Proposition-Definition 2.4.3, we can see that $\langle \varphi, \varphi \rangle \leq 0$ for all $\varphi \in C_{\text{int}}^0(X^{\text{an}})$. Thus $\langle \cdot, \cdot \rangle$ is negative semi-definite. Clearly if φ is a constant function, then $\langle \varphi, \varphi \rangle = 0$. Conversely, we assume $\langle \varphi, \varphi \rangle = 0$. Then, by [27, Lemma 1.1.3], $\langle \psi, \varphi \rangle = 0$ for all $\psi \in C_{\text{int}}^0(X^{\text{an}})$.

Let \mathcal{X} be a regular model of X over $\text{Spec}(k^\circ)$ and $\mathcal{X}_\circ = a_1 C_1 + \cdots + a_r C_r$ the irreducible decomposition of the central fiber \mathcal{X}_\circ as a cycle. Let x_j be the point of X^{an} corresponding to C_j . For $i = 1, \dots, r$, if $\psi_i = g_{(\mathcal{X}, C_i)}$, then

$$\begin{aligned} \deg \left(C_i \cdot \sum_{j=1}^r \varphi(x_j) a_j C_j \right) &= -2 \log v(\varpi) \sum_{j=1}^r \frac{\varphi(x_j) a_j}{-2 \log v(\varpi)} \deg(C_i \cdot C_j) \\ &= -2 \log v(\varpi) \langle \psi_i, \varphi \rangle = 0. \end{aligned}$$

Therefore, by the equality condition of Zariski's lemma (cf. [27, Lemma 1.1.4]), we have

$$\varphi(x_1) = \cdots = \varphi(x_r).$$

Let $X_{\text{div}}^{\text{an}}$ be a subset of X^{an} consisting of valuations arising from irreducible components of the central fiber of any regular model of X . The above observation shows that $\varphi|_{X_{\text{div}}^{\text{an}}}$ is a constant function a . We set $\lambda := \varphi - a$ on X^{an} . By the density theorem, for any positive number ϵ , there is $\theta \in \mathbb{M}(X^{\text{an}}) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $|\lambda - \theta| \leq \epsilon$. Thus $|\theta(x)| \leq \epsilon$ for all $x \in X_{\text{div}}^{\text{an}}$, which implies that $|\theta| \leq \epsilon$ on X^{an} . Indeed, if θ is given by a vertical \mathbb{R} -Cartier divisor $\Theta = c_1 C_1 + \cdots + c_r C_r$ on some regular model \mathcal{X} as before (i.e. $\theta = g_{(\mathcal{X}, \Theta)}$), then $c_i = a_i \theta(x_i) / (-2 \log v(\varpi))$, so that

$$-(\epsilon / (-2 \log v(\varpi))) \mathcal{X}_\circ \leq \Theta \leq (\epsilon / (-2 \log v(\varpi))) \mathcal{X}_\circ,$$

which means that $|\theta| \leq \epsilon$ on X^{an} . Thus $|\lambda| \leq 2\epsilon$, and hence $\lambda = 0$. \square

3. LOCAL AND GLOBAL DENSITY THEOREMS

The density theorem in terms of global model functions was established by Gubler [12, Theorem 7.12] and X. Yuan [30, Lemma 3.5]. Recently an elementary proof was found by Boucksom, Favre and Jonsson [6]. Unfortunately, they assume that the characteristic of the residue field is zero, which is a strong restriction for our purpose. In this section, we will give a proof of the density theorem in general by using their ideas.

Let k be a field and v a non-trivial discrete valuation of k . Note that v is multiplicative. Let ϖ be a uniformizing parameter of k° , that is, $k^\circ = \varpi k^\circ$ (for the definition of k° and k° , see Conventions and terminology 0.5.2). Let X be a projective and geometrically integral variety over k and let $\text{Rat}(X)$ be the rational function field of X .

3.1. Vertical fractional ideal sheaves and birational system of models. Let \mathcal{X} be a model of X over $\text{Spec}(k^\circ)$ (cf. Conventions and terminology 0.5.5). The central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ is denoted by \mathcal{X}_\circ , that is, $\mathcal{X}_\circ := \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k^\circ / k^\circ)$. A non-zero coherent subsheaf \mathcal{J} of $\text{Rat}(X)$ on \mathcal{X} is called a *fractional ideal sheaf* on \mathcal{X} . It is said to be *vertical* if there is $m \in \mathbb{Z}_{\geq 0}$ such that $\varpi^m \mathcal{J}$ is an ideal sheaf and $\text{Supp}(\mathcal{O}_{\mathcal{X}} / \varpi^m \mathcal{J}) \subseteq \mathcal{X}_\circ$. Let \mathcal{D} be a vertical Cartier divisor on \mathcal{X} , that is, $\text{Supp}_{\mathbb{Z}}(\mathcal{D}) \subseteq \mathcal{X}_\circ$ (for the definition of $\text{Supp}_{\mathbb{Z}}$, see Subsection 1.1). Note that $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is a vertical fractional sheaf. Indeed, let $\xi \in \mathcal{X}_\circ$ and f a local equation of \mathcal{D} at ξ . Then f is a unit element of $((\mathcal{O}_{\mathcal{X}, \xi})_S)_p$ for all $p \in \text{Spec}((\mathcal{O}_{\mathcal{X}, \xi})_S)$, where S is a multiplicative set given by $\{1, \varpi, \varpi^2, \dots\}$, and hence

$f \in (\mathcal{O}_{\mathcal{X}, \xi})_S$. Thus we can find $m_\xi \in \mathbb{Z}_{\geq 0}$ such that $\varpi^{m_\xi} f \in \mathcal{O}_{\mathcal{X}, \xi}$. This observation shows that $\varpi^m \mathcal{O}_{\mathcal{X}}(\mathcal{D}) \subseteq \mathcal{O}_{\mathcal{X}}$ for some $m \in \mathbb{Z}_{\geq 0}$, as required.

A set Ψ of models of X is called a *birational system of models of X* if the following conditions are satisfied:

- (1) For any $\mathcal{X} \in \Psi$ and any vertical fractional ideal sheaf \mathcal{J} on \mathcal{X} , there is $\mathcal{X}' \in \Psi$ together with a birational morphism $\nu : \mathcal{X}' \rightarrow \mathcal{X}$ such that $\mathcal{J} \mathcal{O}_{\mathcal{X}'}$ is invertible.
- (2) For any $\mathcal{X}_1, \mathcal{X}_2 \in \Psi$, there is $\mathcal{X}_3 \in \Psi$ together with birational morphisms $\nu_1 : \mathcal{X}_3 \rightarrow \mathcal{X}_1$ and $\nu_2 : \mathcal{X}_3 \rightarrow \mathcal{X}_2$.

Remark 3.1.1. The set of all models of X is a birational system of models of X . In addition, fixing a model \mathcal{Z} of X , the set of all models \mathcal{X} over \mathcal{Z} (that is, there is a birational morphism $\mathcal{X} \rightarrow \mathcal{Z}$) forms a birational system of models of X .

3.2. Model functions. We assume that ν is complete. Let $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ be a model of X . Let \mathcal{J} be a vertical fractional ideal sheaf on \mathcal{X} . According to [6, Subsection 2.3], we define a function $\log|\mathcal{J}|$ on X^{an} to be

$$\log|\mathcal{J}|(x) := \log \max\{|h(x)| \mid h \in \mathcal{J}_{r_{\mathcal{X}}(x)}\},$$

where $r_{\mathcal{X}} : X^{\text{an}} \rightarrow \mathcal{X}_\circ$ is the reduction map induced by the model \mathcal{X} . For example, $\log|\varpi \mathcal{O}_{\mathcal{X}}| = \log \nu(\varpi)$. Note that $\log|\mathcal{J}| = \log|\mathcal{J} \mathcal{O}_{\mathcal{X}'}|$ for a birational morphism $\nu : \mathcal{X}' \rightarrow \mathcal{X}$ of models of X and a vertical fractional ideal sheaf \mathcal{J} on \mathcal{X} . Moreover, if \mathcal{J} is invertible, then $\log|\mathcal{J}| = g_{(\mathcal{X}, \mathcal{J})}/2$ (cf. Subsection 2.1). Let us fix a birational system Ψ of models of X . A function φ on X^{an} is called a *model function with respect to Ψ* if there are $\mathcal{X} \in \Psi$ and a vertical fractional ideal sheaf \mathcal{J} on \mathcal{X} such that $\varphi = \log|\mathcal{J}|$. The set of all model functions with respect to Ψ is denoted by $\mathbb{M}(X^{\text{an}}; \Psi)$. Then we have the following generalization of [6, Proposition 2.2].

Proposition 3.2.1. (1) For $\varphi \in \mathbb{M}(X^{\text{an}}; \Psi)$, there are $\mathcal{X} \in \Psi$ and a vertical Cartier divisor \mathcal{D} on \mathcal{X} such that $\varphi = \log|\mathcal{O}_{\mathcal{X}}(\mathcal{D})|$. In particular, $\mathbb{M}(X^{\text{an}}; \Psi)$ forms an abelian group and $\mathbb{M}(X^{\text{an}}; \Psi) \subseteq C^0(X^{\text{an}})$ (cf. Proposition 2.1.4).

(2) For $\varphi_1, \varphi_2 \in \mathbb{M}(X^{\text{an}}; \Psi)$, $\max\{\varphi_1, \varphi_2\} \in \mathbb{M}(X^{\text{an}}; \Psi)$.

(3) For $x, y \in X^{\text{an}}$ with $x \neq y$, there is $\varphi \in \mathbb{M}(X^{\text{an}}; \Psi)$ such that $\varphi(x) \neq \varphi(y)$.

Proof. The assertions of the proposition can be proved by the ideas of [6, Proposition 2.2].

(1) We choose $\mathcal{X} \in \Psi$ and a vertical fractional ideal sheaf \mathcal{J} on \mathcal{X} such that $\varphi = \log|\mathcal{J}|$. By our assumption, there is $\mathcal{X}' \in \Psi$ together with a birational morphism $\nu : \mathcal{X}' \rightarrow \mathcal{X}$ such that $\mathcal{J} \mathcal{O}_{\mathcal{X}'}$ is invertible, that is, there is a vertical Cartier divisor \mathcal{D}' on \mathcal{X}' such that $\mathcal{J} \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(\mathcal{D}')$. Clearly $\log|\mathcal{J}| = \log|\mathcal{O}_{\mathcal{X}'}(\mathcal{D}')|$, as desired.

(2) By (1) and the property of Ψ , we can find $\mathcal{X} \in \Psi$ and vertical Cartier divisors \mathcal{D}_1 and \mathcal{D}_2 on \mathcal{X} such that $\varphi_1 = \log|\mathcal{O}_{\mathcal{X}}(\mathcal{D}_1)|$ and $\varphi_2 = \log|\mathcal{O}_{\mathcal{X}}(\mathcal{D}_2)|$. If we set $\mathcal{J} = \mathcal{O}_{\mathcal{X}}(\mathcal{D}_1) + \mathcal{O}_{\mathcal{X}}(\mathcal{D}_2)$ in $\text{Rat}(\mathcal{X})$, then \mathcal{J} is a vertical fractional ideal sheaf of \mathcal{X} and $\log|\mathcal{J}| = \max\{\varphi_1, \varphi_2\}$.

(3) Fix $\mathcal{X} \in \Psi$. First we assume that $r_{\mathcal{X}}(x) \neq r_{\mathcal{X}}(y)$. Let \mathfrak{m} be the maximal ideal at $r_{\mathcal{X}}(x)$. Then $\log|\mathfrak{m}|(x) < 0$ and $\log|\mathfrak{m}|(y) \geq 0$, as desired. Next we assume that $\xi = r_{\mathcal{X}}(x) = r_{\mathcal{X}}(y)$. Let $\mathcal{U} = \text{Spec}(\mathcal{A})$ be an affine open neighborhood of ξ . We can find $f \in \mathcal{A}$ such that $|f|_x \neq |f|_y$. Since \mathcal{X} is noetherian, there is an ideal sheaf \mathcal{I} on \mathcal{X} such that $\mathcal{I}_\xi = f \mathcal{O}_{\mathcal{X}, \xi}$. For each $m \in \mathbb{Z}_{>0}$, we set $\mathcal{J}_m = \mathcal{I} + \varpi^m \mathcal{O}_{\mathcal{X}}$. Note that \mathcal{J}_m is a vertical

ideal sheaf on \mathcal{X} . Moreover,

$$\begin{cases} \log |\mathcal{I}_m|(x) = \max\{\log |f|_x, m \log v(\varpi)\}, \\ \log |\mathcal{I}_m|(y) = \max\{\log |f|_y, m \log v(\varpi)\}, \end{cases}$$

where $\log(0) = -\infty$. As $|f|_x \neq |f|_y$ and $v(\varpi) < 1$, if m is sufficiently large, then

$$\log |\mathcal{I}_m|(x) \neq \log |\mathcal{I}_m|(y),$$

as required. \square

3.3. Density theorems. Let k_v be the completion of k with respect to v . By abuse of notation, the unique extension of v to k_v is also denoted by v . We set $X_v = X \times_{\text{Spec}(k)} \text{Spec}(k_v)$, which is also a projective and geometrically integral variety over k_v . For a model $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ of X , $\mathcal{X}_v := \mathcal{X} \times_{\text{Spec}(k^\circ)} \text{Spec}(k_v^\circ)$ is also a model of X_v by (1.1) in Lemma 1.3.7. The projection $\mathcal{X}_v \rightarrow \mathcal{X}$ is denoted by $\pi_{\mathcal{X}}$. Let Ψ be a system of models of X .

Proposition 3.3.1. $\Psi_v := \{\mathcal{X}_v \mid \mathcal{X} \in \Psi\}$ forms a system of models of X_v .

Proof. The assertion follows from (2) in the following lemma. \square

Lemma 3.3.2. (1) Let \mathcal{I}_v be a vertical ideal sheaf on \mathcal{X}_v . Then there is $n \in \mathbb{Z}_{\geq 0}$ such that $\varpi^n \mathcal{O}_{\mathcal{X}_v} \subseteq \mathcal{I}_v$.

(2) Let \mathcal{I}_v be a vertical fractional ideal sheaf on \mathcal{X}_v . Then there is a vertical fractional ideal sheaf \mathcal{I} on \mathcal{X} such that $\mathcal{I} \mathcal{O}_{\mathcal{X}_v} = \mathcal{I}_v$.

Proof. (1) For $\xi \in (\mathcal{X}_v)_\circ$, $(\mathcal{O}_{\mathcal{X}_v, \xi} / (\mathcal{I}_v)_\xi)_S = 0$ because $\text{Supp}(\mathcal{O}_{\mathcal{X}_v} / \mathcal{I}_v) \subseteq (\mathcal{X}_v)_\circ$, where S is a multiplicative set given by $\{1, \varpi, \varpi^2, \dots\}$. Therefore,

$$\varpi^{m_\xi} (\mathcal{O}_{\mathcal{X}_v, \xi} / (\mathcal{I}_v)_\xi) = 0$$

for some $m_\xi \in \mathbb{Z}_{\geq 0}$, and hence $\varpi^{m_\xi} \mathcal{O}_{\mathcal{X}_v, \xi} \subseteq (\mathcal{I}_v)_\xi$. Thus the assertion follows.

(2) Clearly we may assume that \mathcal{I}_v is an ideal sheaf. Then, by (1), there is $n \in \mathbb{Z}_{\geq 0}$ such that $\varpi^n \mathcal{O}_{\mathcal{X}_v} \subseteq \mathcal{I}_v$. As

$$\mathcal{I}_v / \varpi^n \mathcal{O}_{\mathcal{X}_v} \subseteq \mathcal{O}_{\mathcal{X}_v} / \varpi^n \mathcal{O}_{\mathcal{X}_v} \xleftarrow[\pi_{\mathcal{X}}^*]{\sim} \mathcal{O}_{\mathcal{X}} / \varpi^n \mathcal{O}_{\mathcal{X}},$$

we can find an ideal sheaf \mathcal{I} on \mathcal{X} such that $\varpi^n \mathcal{O}_{\mathcal{X}} \subseteq \mathcal{I}$ and $\mathcal{I}_v / \varpi^n \mathcal{O}_{\mathcal{X}_v} \simeq \mathcal{I} / \varpi^n \mathcal{O}_{\mathcal{X}}$, so that we can easily see that $\mathcal{I} \mathcal{O}_{\mathcal{X}_v} = \mathcal{I}_v$. \square

Theorem 3.3.3 (Local density theorem). $\mathbb{M}(X_v^{\text{an}}; \Psi_v) \otimes_{\mathbb{Z}} \mathbb{Q}$ is dense in $C^0(X_v^{\text{an}})$ with respect to the supremum norm $\|\cdot\|_{\text{sup}}$.

Proof. We set $T = X_v^{\text{an}}$ and $\Sigma = \mathbb{M}(X_v^{\text{an}}; \Psi_v) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is well known that T is a compact Hausdorff space (cf. [2, Theorem 3.4.8]). Thus, if we can check the conditions (1) – (4) in Lemma 3.3.4, we have the assertion.

(1) follows from (3) in Proposition 3.2.1.

(2) Note that $\log v(\varpi) \in \mathbb{M}(X_v^{\text{an}}; \Psi_v) \subseteq \Sigma$.

(3) is obvious.

(4) Let us check that $\max\{\psi_1, \psi_2\} \in \Sigma$ for $\psi_1, \psi_2 \in \Sigma$. We choose $n \in \mathbb{Z}_{>0}$ such that $n\varphi_1, n\varphi_2 \in \mathbb{M}(X_v^{\text{an}}; \Psi_v)$. Then, by (2) in Proposition 3.2.1,

$$n \max\{\psi_1, \psi_2\} = \max\{n\psi_1, n\psi_2\} \in \Sigma,$$

and hence $\max\{\psi_1, \psi_2\} \in \Sigma$. \square

Lemma 3.3.4. *Let T be a compact Hausdorff space. Let Σ be a subset of $C^0(T)$ with the following properties:*

- (1) *For any $x, y \in T$ with $x \neq y$, there is $f \in \Sigma$ such that $f(x) \neq f(y)$.*
- (2) *$\mathbb{R}^\times \cap \Sigma \neq \emptyset$.*
- (3) *Σ forms a \mathbb{Q} -vector space.*
- (4) *For $f, g \in \Sigma$, $\max\{f, g\} \in \Sigma$.*

Then Σ is dense in $C^0(T)$ with respect to the supremum norm $\|\cdot\|_{\sup}$.

Proof. Let $\bar{\Sigma}$ be the closure of Σ with respect to $\|\cdot\|_{\sup}$. Then it is easy to see that $\bar{\Sigma}$ has the following properties:

- (1)' *For any $x, y \in T$ with $x \neq y$, there is $f \in \bar{\Sigma}$ such that $f(x) \neq f(y)$.*
- (2)' *$1 \in \bar{\Sigma}$.*
- (3)' *$\bar{\Sigma}$ forms an \mathbb{R} -vector space.*
- (4)' *For $f, g \in \bar{\Sigma}$, $\max\{f, g\} \in \bar{\Sigma}$.*

Thus, by [15, Theorem 7.29], $\bar{\Sigma}$ is dense in $C^0(T)$. Note that $\bar{\Sigma} = \overline{\bar{\Sigma}}$. Therefore the assertion follows. \square

Definition 3.3.5. A function φ on X_v^{an} is called a *global model function with respect to Ψ* if there are $\mathcal{X} \in \Psi$ and a vertical fractional ideal sheaf \mathcal{J} on \mathcal{X} such that $\varphi = \log|\mathcal{J} \mathcal{O}_{\mathcal{X}_v}|$. The set of all global model functions with respect to Ψ is denoted by $\mathbb{M}(X; \Psi)$.

By using (2) in Lemma 3.3.2, we can see

$$\mathbb{M}(X; \Psi) = \mathbb{M}(X_v^{\text{an}}; \Psi_v),$$

Thus the local density theorem (cf. Theorem 3.3.3) implies the following main result of this section.

Theorem 3.3.6 (Global density theorem). *$\mathbb{M}(X; \Psi) \otimes_{\mathbb{Z}} \mathbb{Q}$ is dense in $C^0(X_v^{\text{an}})$ with respect to the supremum norm $\|\cdot\|_{\sup}$.*

The following theorem is an application of the global density theorem.

Theorem 3.3.7 (Approximation theorem of adelic \mathbb{R} -divisors). *We assume that k° is excellent and X is normal. Let $\bar{D} = (D, g)$ be an adelic \mathbb{R} -Cartier divisor of C^0 -type on X . For any positive number $\epsilon > 0$, there is a normal model \mathcal{X} of X over $\text{Spec}(k^\circ)$, and \mathbb{R} -Cartier divisors \mathcal{D}_1 and \mathcal{D}_2 on \mathcal{X} such that*

$$\bar{D} - (0, \epsilon) \leq \mathcal{D}_1 \leq \bar{D} \leq \mathcal{D}_2 \leq \bar{D} + (0, \epsilon).$$

Proof. Let us begin with the following claim:

Claim 3.3.7.1. *For a positive number ϵ , there is a normal model \mathcal{X} of X over $\text{Spec}(k^\circ)$ and an \mathbb{R} -Cartier divisor \mathcal{D} on \mathcal{X} such that $\mathcal{D} \cap X = D$ and*

$$\|g_{(\mathcal{X}, \mathcal{D})} - g\|_{\sup} \leq \epsilon/2.$$

Proof. First let us choose a model \mathcal{X}_0 of X over $\text{Spec}(k^\circ)$ and an \mathbb{R} -Cartier divisor \mathcal{D}_0 on \mathcal{X}_0 such that $D = \mathcal{D}_0 \cap X$. Let g_0 be the D -Green function of C^0 -type on X_v^{an} arising from the model $(\mathcal{X}_0, \mathcal{D}_0)$. We set $\phi := g - g_0$. Then ϕ is a continuous function on X_v^{an} . Let Ψ be the set of all models of X over \mathcal{X}_0 (cf. Remark 3.1.1). By the global density theorem (cf. Theorem 3.3.6), there is a global model function φ on X_v^{an} with respect to Ψ such that $\|\phi - \varphi\|_{\sup} \leq \epsilon/2$, that is, there are a model \mathcal{X}' in Ψ together with a birational morphism $\mu : \mathcal{X}' \rightarrow \mathcal{X}_0$, a vertical Cartier divisor \mathcal{E}' on \mathcal{X}' and $a \in \mathbb{Q}$ such that $\varphi = a g_{(\mathcal{X}', \mathcal{E}')}.$ Let

$\pi : \mathcal{X} \rightarrow \mathcal{X}'$ be the normalization of \mathcal{X}' and $\mathcal{D} := a\pi^*(\mathcal{E}') + \pi^*(\mu^*(\mathcal{D}_0))$. As k° is an excellent, π is a finite morphism, that is, $\mathcal{X} \in \Psi$. In addition, $g_{(\mathcal{X}, \mathcal{D})} = \varphi + g_0$. Therefore, we have the assertion of the claim. \square

We set $g_1 = g_{(\mathcal{X}, \mathcal{D})} - \epsilon/2$ and $g_2 = g_{(\mathcal{X}, \mathcal{D})} + \epsilon/2$. Then

$$g - \epsilon \leq g_1 \leq g \leq g_2 \leq g + \epsilon$$

on X_v^{an} . Moreover, note that the global model function arising from the central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ is a constant function. Thus the assertion follows. \square

4. ADELIC ARITHMETIC \mathbb{R} -CARTIER DIVISORS

In this section, we will introduce adelic arithmetic \mathbb{R} -Cartier divisors on arithmetic varieties and investigate their several basic properties.

Throughout this section, let K be a number field and let X be a d -dimensional, projective, smooth and geometrically integral variety over K .

4.1. Definition and basic properties. Let O_K be the ring of integers in K . We denote the set of all maximal ideals of O_K by M_K . For $P \in M_K$, the valuation v_P of K at P is given by

$$v_P(x) = \#(O_K/P)^{-\text{ord}_P(x)}.$$

Let K_P be the completion of K with respect to v_P and let $X_P := X \times_{\text{Spec}(K)} \text{Spec}(K_P)$, which is also a projective, smooth and geometrically integral variety over K_P . Let $X(\mathbb{C})$ be the set of all \mathbb{C} -valued points of X , that is,

$$X(\mathbb{C}) := \{x : \text{Spec}(\mathbb{C}) \rightarrow X \mid x \text{ is a morphism as schemes}\}.$$

Note that $X(\mathbb{C})$ is a projective complex manifold and $X(\mathbb{C})$ is not necessarily connected. Let $F_\infty : X(\mathbb{C}) \rightarrow X(\mathbb{C})$ be the complex conjugation map, that is, for $x \in X(\mathbb{C})$, $F_\infty(x)$ is given by the composition of morphisms

$$\text{Spec}(\mathbb{C}) \xrightarrow{-a} \text{Spec}(\mathbb{C}) \quad \text{and} \quad \text{Spec}(\mathbb{C}) \xrightarrow{x} X,$$

where $\text{Spec}(\mathbb{C}) \xrightarrow{-a} \text{Spec}(\mathbb{C})$ is the morphism arising from the complex conjugation. The complex conjugation map $F_\infty : X(\mathbb{C}) \rightarrow X(\mathbb{C})$ is an anti-holomorphic isomorphism. The space of F_∞ -invariant real valued continuous functions on $X(\mathbb{C})$ is denoted by $C_{F_\infty}^0(X(\mathbb{C}))$, that is,

$$C_{F_\infty}^0(X(\mathbb{C})) := \{f \in C^0(X(\mathbb{C})) \mid f \circ F_\infty = f\}.$$

Definition 4.1.1. A pair $\overline{D} = (D, g)$ of an \mathbb{R} -Cartier divisor D on X and a collection of Green functions $g = \{g_P\}_{P \in M_K} \cup \{g_\infty\}$ is called an *adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X* if the following conditions (1) and (2) hold:

- (1) For all $P \in M_K$, g_P is a D -Green function of C^0 -type on X_P^{an} . In addition, there exist a non-empty open set U of $\text{Spec}(O_K)$, a normal model \mathcal{X}_U of X over U and an \mathbb{R} -Cartier divisor \mathcal{D}_U on \mathcal{X}_U such that $\mathcal{D}_U \cap X = D$ and g_P is a D -Green function induced by $(\mathcal{X}_U, \mathcal{D}_U)$ for all $P \in U \cap M_K$.
- (2) The Green function g_∞ is a D -Green function of C^0 -type on $X(\mathbb{C})$ such that $g_\infty = g_\infty \circ F_\infty$ (cf. [26, Section 5]).

Moreover, a pair $\overline{D}' = (D, g')$ of an \mathbb{R} -Cartier divisor D on X and a collection of Green functions $g' = \{g'_P\}_{P \in M_K}$ is called a *global adelic \mathbb{R} -Cartier divisor of C^0 -type on X* if the above condition (1) holds for $\{g'_P\}_{P \in M_K}$, that is, g'_P is a D -Green function of C^0 -type on X_P^{an} for all $P \in M_K$, and there exist a non-empty open set U' of $\text{Spec}(O_K)$, a normal model $\mathcal{X}'_{U'}$ of X over U' and an \mathbb{R} -Cartier divisor $\mathcal{D}'_{U'}$ on $\mathcal{X}'_{U'}$ such that $\mathcal{D}'_{U'} \cap X = D$ and g'_P is a D -Green function induced by $(\mathcal{X}'_{U'}, \mathcal{D}'_{U'})$ for all $P \in U' \cap M_K$.

The pair $(\mathcal{X}_U, \mathcal{D}_U)$ in the condition (1) is called a *defining model of \overline{D} over U* . If we forget the Green function g_∞ on $X(\mathbb{C})$ from \overline{D} , we have a global adelic \mathbb{R} -Cartier divisor on X , which is denoted by \overline{D}^τ and is called the *truncation of \overline{D}* . For simplicity, a collection of Green functions $g = \{g_P\}_{P \in M_K} \cup \{g_\infty\}$ is often denoted by the following symbol:

$$g = \sum_{P \in M_K} g_P[P] + g_\infty[\infty].$$

Let $\text{Rat}(X)$ be the rational function field of X . For $\varphi \in \text{Rat}(X)^\times$, we define $(\widehat{\varphi})$ to be

$$(\widehat{\varphi}) := \left((\varphi), \sum_{P \in M_K} (-\log|\varphi_P|^2)[P] + (-\log|\varphi_\infty|^2)[\infty] \right),$$

where φ_P and φ_∞ are the rational functions on X_P^{an} and $X(\mathbb{C})$ induced by φ , respectively. The adelic arithmetic divisor $(\widehat{\varphi})$ is called an *adelic arithmetic principal divisor*. Let $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$ be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . For $a_1, a_2 \in \mathbb{R}$, we define $a_1\overline{D}_1 + a_2\overline{D}_2$ to be

$$a_1\overline{D}_1 + a_2\overline{D}_2 := (a_1D_1 + a_2D_2, a_1g_1 + a_2g_2),$$

where $a_1g_1 + a_2g_2 = \sum_{P \in M_K} (a_1(g_1)_P + a_2(g_2)_P)[P] + (a_1(g_1)_\infty + a_2(g_2)_\infty)[\infty]$. The space of all adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X is denoted by $\widehat{\text{Div}}_{C^0}^a(X)_\mathbb{R}$, which forms an \mathbb{R} -vector space by the previous definition. Moreover, we define $\overline{D}_1 \leq \overline{D}_2$ by the following conditions:

- (a) $D_1 \leq D_2$.
- (b) $(g_1)_P \leq (g_2)_P$ for all $P \in M_K$ and $(g_1)_\infty \leq (g_2)_\infty$.

Similarly, for global adelic \mathbb{R} -Cartier divisors

$$(D_1, \{(g_1)_P\}_{P \in M_K}) \quad \text{and} \quad (D_2, \{(g_2)_P\}_{P \in M_K}),$$

$$(D_1, \{(g_1)_P\}_{P \in M_K}) \leq (D_2, \{(g_2)_P\}_{P \in M_K})$$

is defined by $D_1 \leq D_2$ and $(g_1)_P \leq (g_2)_P$ for all $P \in M_K$.

Let \mathcal{X} be a normal model of X over $\text{Spec}(O_K)$ and let $\overline{\mathcal{D}} = (\mathcal{D}, g_\infty)$ be an arithmetic \mathbb{R} -Cartier divisor of C^0 -type on \mathcal{X} (cf. [26, Section 5]). The pair $(\mathcal{X}, \overline{\mathcal{D}})$ gives rise to an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X , that is,

$$\left(\mathcal{D} \cap X, \sum_{P \in M_K} g_{(\mathcal{X}_{(P)}, \mathcal{D}_{(P)})}[P] + g_\infty[\infty] \right),$$

where $\mathcal{X}_{(P)}$ is the localization of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ at P and $\mathcal{D}_{(P)}$ is the resection of \mathcal{D} to $\mathcal{X}_{(P)}$. We use the symbol $\mathcal{X}_{(P)}$ to distinguish it from X_P at the beginning of this subsection. We denote it by $\overline{\mathcal{D}}^a$ and it is called the *associated adelic arithmetic \mathbb{R} -Cartier divisor with $\overline{\mathcal{D}}$* . Note that $(\widehat{\varphi}) = \left((\widehat{\varphi})_{\mathcal{X}} \right)^a$ for $\varphi \in \text{Rat}(\mathcal{X})^\times$, where $(\widehat{\varphi})_{\mathcal{X}}$ is the arithmetic principal

divisor of φ on \mathcal{X} . Similarly, the *associated global adelic \mathbb{R} -Cartier divisor \mathcal{D}^a with \mathcal{D}* is defined by

$$\mathcal{D}^a := \left(\mathcal{D} \cap X, \left\{ g_{(\mathcal{X}(P), \mathcal{D}(P))} \right\}_{P \in M_K} \right).$$

By abuse of notation, we often use the notations $\overline{\mathcal{D}} \leq \overline{\mathcal{D}}_2$ and $\overline{\mathcal{D}}_1 \leq \overline{\mathcal{D}}$ instead of $\overline{\mathcal{D}}^a \leq \overline{\mathcal{D}}_2$ and $\overline{\mathcal{D}}_1 \leq \overline{\mathcal{D}}^a$ respectively. The following proposition is the arithmetic version of Proposition 2.2.1 and it follows from Proposition 2.2.1.

Proposition 4.1.2. *Let \mathcal{X} be a normal model of X over $\text{Spec}(O_K)$ and let $\widehat{\text{Div}}_{C^0}(\mathcal{X})_{\mathbb{R}}$ be the space of arithmetic \mathbb{R} -Cartier divisors of C^0 -type on \mathcal{X} . Let*

$$\iota : \widehat{\text{Div}}_{C^0}(\mathcal{X})_{\mathbb{R}} \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$$

be the map given by $\overline{\mathcal{D}} \mapsto \overline{\mathcal{D}}^a$. Then we have the following:

- (1) *The map $\iota : \widehat{\text{Div}}_{C^0}(\mathcal{X})_{\mathbb{R}} \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$ is an injective homomorphism of \mathbb{R} -vector spaces.*
- (2) *$\overline{\mathcal{D}}_1 \leq \overline{\mathcal{D}}_2 \iff \overline{\mathcal{D}}_1^a \leq \overline{\mathcal{D}}_2^a$.*

The following theorem is a consequence of Theorem 3.3.7.

Theorem 4.1.3 (Approximation theorem of adelic arithmetic \mathbb{R} -divisors). *Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X and let $(\mathcal{X}_U, \mathcal{D}_U)$ be a defining model of \overline{D} over a non-empty open set $U \subseteq \text{Spec}(O_K)$. For any positive number $\epsilon > 0$, there exist a normal model \mathcal{X}_ϵ over $\text{Spec}(O_K)$, and \mathbb{R} -Cartier divisors \mathcal{D}_1 and \mathcal{D}_2 on \mathcal{X}_ϵ with the following properties:*

- (1) *$\mathcal{X}_\epsilon|_U = \mathcal{X}_U$, $\mathcal{D}_1|_U = \mathcal{D}_U$ and $\mathcal{D}_2|_U = \mathcal{D}_U$.*
- (2) *If we set $S = M_K \setminus U$, $\overline{\mathcal{D}}_1 = (\mathcal{D}_1, g_\infty)$ and $\overline{\mathcal{D}}_2 = (\mathcal{D}_2, g_\infty)$, then*

$$\overline{D} - \left(0, \sum_{P \in S} \epsilon[P] \right) \leq \overline{\mathcal{D}}_1^a \leq \overline{D} \leq \overline{\mathcal{D}}_2^a \leq \overline{D} + \left(0, \sum_{P \in S} \epsilon[P] \right).$$

4.2. Global degree. Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Let x be a closed point of X . First we assume that $x \notin \text{Supp}_{\mathbb{R}}(D)$. For $P \in M_K$, the local degree of \overline{D} over the valuation v_P is denoted by $\widehat{\deg}_P(\overline{D}|_x)$ (cf. Subsection 2.3). Moreover, $\widehat{\deg}_{\infty}(\overline{D}|_x)$ is defined by

$$\widehat{\deg}_{\infty}(\overline{D}|_x) := \frac{1}{2} \sum_{\sigma: K(x) \rightarrow \mathbb{C}} g_{\infty}(x_{\sigma}),$$

where $K(x)$ is the residue field at x and x_{σ} is the \mathbb{C} -value point given by

$$\mathcal{O}_{X,x} \rightarrow K(x) \xrightarrow{\sigma} \mathbb{C}.$$

Let U be a non-empty Zariski open set of $\text{Spec}(O_K)$ such that \overline{D} has a defining model $(\mathcal{X}_U, \mathcal{D}_U)$ over U . Let Δ_x be the closure of x in \mathcal{X}_U . Shrinking U if necessarily, we may assume that $\Delta_x \cap \text{Supp}(\mathcal{D}_U) = \emptyset$, which implies that $\widehat{\deg}_P(\overline{D}|_x) = 0$ for $P \in U$. Therefore, we can define $\widehat{\deg}(\overline{D}|_x)$ to be

$$\widehat{\deg}(\overline{D}|_x) = \sum_{P \in M_K} \widehat{\deg}_P(\overline{D}|_x) + \widehat{\deg}_{\infty}(\overline{D}|_x).$$

Note that

$$(4.2.1) \quad \widehat{\deg}(\widehat{(\varphi)}|_x) = 0$$

for $\varphi \in \text{Rat}(X)^\times$ with $x \notin \text{Supp}((\varphi))$. In general, we can find $\phi \in \text{Rat}(X)_{\mathbb{R}}^\times$ such that $x \notin \text{Supp}(D + (\phi))$ (cf. [26, Lemma 5.2.3]). By using (4.2.1), we can see that the quantity $\widehat{\deg}(\overline{D} + (\widehat{\phi})|_x)$ does not depend on the choice of ϕ , so that it is denoted by $\widehat{\deg}(\overline{D}|_x)$ and is called the *global degree of \overline{D} along x* . The equation (4.2.1) can be generalized as follows:

$$(4.2.2) \quad \widehat{\deg}((\widehat{\psi})|_x) = 0$$

for all $\psi \in \text{Rat}(X)^\times$.

Lemma 4.2.3. *Let $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$ be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . If $D_1 = D_2$ and $g_1 \leq g_2$, then $\widehat{\deg}(\overline{D}_1|_x) \leq \widehat{\deg}(\overline{D}_2|_x)$ for all closed points x of X .*

Proof. As $D_1 = D_2$ and $g_1 \leq g_2$, there are non-negative continuous functions ϕ_P on X_P^{an} and ϕ_∞ on $X(\mathbb{C})$ such that $(g_2)_P = (g_1)_P + \phi_P$ and $(g_2)_\infty = (g_1)_\infty + \phi_\infty$, respectively. We set $\phi = \sum_{P \in M_K} \phi_P[P] + \phi_\infty[\infty]$. Then, as $\widehat{\deg}((0, \phi)|_x) \geq 0$,

$$\widehat{\deg}(\overline{D}_2|_x) = \widehat{\deg}(\overline{D}_1|_x) + \widehat{\deg}((0, \phi)|_x) \geq \widehat{\deg}(\overline{D}_1|_x),$$

as required. \square

4.3. Volume of adelic arithmetic \mathbb{R} -Cartier divisors. Let D be an \mathbb{R} -Cartier divisor on X , $\overline{D}' = (D, g')$ a global adelic \mathbb{R} -Cartier divisor of C^0 -type on X , and $\overline{D} = (D, g)$ an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . We define $H^0(X, D)$, $\hat{H}^0(X, \overline{D}')$ and $\hat{H}^0(X, \overline{D})$ to be

$$\begin{cases} H^0(X, D) := \{\varphi \in \text{Rat}(X)^\times \mid D + (\varphi) \geq 0\} \cup \{0\}, \\ \hat{H}^0(X, \overline{D}') := \{\varphi \in H^0(X, D) \mid \|\varphi\|_{g'_P} \leq 1 \text{ for all } P \in M_K\}, \\ \hat{H}^0(X, \overline{D}) := \{\varphi \in H^0(X, D) \mid \|\varphi\|_{g_P} \leq 1 \text{ for all } P \in M_K \cup \{\infty\}\}. \end{cases}$$

Note that $\hat{H}^0(X, \overline{D}')$ is a submodule of $H^0(X, D)$ by using Proposition 2.1.3. Let us check the following proposition:

Proposition 4.3.1. (1) $\hat{H}^0(X, \overline{D}')$ and $\hat{H}^0(X, \overline{D})$ are given in the following ways:

$$\begin{cases} \hat{H}^0(X, \overline{D}') = \{\varphi \in \text{Rat}(X)^\times \mid \overline{D}' + (\widehat{\varphi})^\tau \geq 0\} \cup \{0\}, \\ \hat{H}^0(X, \overline{D}) = \{\varphi \in \text{Rat}(X)^\times \mid \overline{D} + (\widehat{\varphi}) \geq 0\} \cup \{0\}. \end{cases}$$

(2) We assume that $\overline{D}^\tau = \overline{D}'$. If there are a normal model \mathcal{X} of X over $\text{Spec}(O_K)$ and an \mathbb{R} -Cartier divisor \mathcal{D} on \mathcal{X} such that $\mathcal{D} \cap X = D$ and g_P is the Green function arising from $(\mathcal{X}, \mathcal{D})$ for each $P \in M_K$, then

$$\hat{H}^0(X, \overline{D}') = H^0(\mathcal{X}, \mathcal{D}) \quad \text{and} \quad \hat{H}^0(X, \overline{D}) = \hat{H}^0(\mathcal{X}, (\mathcal{D}, g_\infty)).$$

(3) $\hat{H}^0(X, \overline{D}')$ is a finitely generated free \mathbb{Z} -module and $\hat{H}^0(X, \overline{D})$ is a finite set. We denote $\log \#(\hat{H}^0(X, \overline{D}))$ by $\hat{h}^0(X, \overline{D})$.

Proof. (1) Note that

$$\|\varphi\|_{g_\infty} \leq 1 \iff g_\infty - \log |\varphi|^2 \geq 0 \text{ on } X(\mathbb{C})$$

and

$$\|\varphi\|_{g_P} \leq 1 \iff g_P - \log |\varphi|^2 \geq 0 \text{ on } X_P^{\text{an}}$$

for $P \in M_K$. Thus (1) follows.

(2) The assertions of (2) follow from (1) and Proposition 4.1.2.

(3) Clearly we may assume that $\overline{D}^\tau = \overline{D}'$. We can find a normal model \mathcal{X} of X over $\text{Spec}(O_K)$ and an arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}} = (\mathcal{D}, h)$ of C^0 -type on \mathcal{X} such that $\overline{D} \leq \overline{\mathcal{D}}^a$. Thus

$$\hat{H}^0(X, \overline{D}') \subseteq \hat{H}^0(X, \overline{\mathcal{D}}^a) = H^0(\mathcal{X}, \mathcal{D})$$

by (2). Note that $H^0(\mathcal{X}, \mathcal{D})$ is a finitely generated free \mathbb{Z} -module, so that $\hat{H}^0(X, \overline{D}')$ is also a finitely generated free \mathbb{Z} -module. Since $\hat{H}^0(X, \overline{D}) \subseteq \hat{H}^0(X, \overline{\mathcal{D}})$, the last assertion is obvious. \square

Definition 4.3.2. Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . The quantity $\hat{\chi}(X, \overline{D})$ is defined by

$$\hat{\chi}(X, \overline{D}) := \hat{\chi}(\hat{H}^0(X, \overline{D}^\tau), \|\cdot\|_{g_\infty})$$

(cf. Conventions and terminology 0.5.3). Note that

$$\hat{h}^0(X, \overline{D}) = \hat{h}^0(\hat{H}^0(X, \overline{D}^\tau), \|\cdot\|_{g_\infty}).$$

Moreover, we define the *volume* $\widehat{\text{vol}}(\overline{D})$ of \overline{D} and the $\hat{\chi}$ -*volume* $\widehat{\text{vol}}_\chi(\overline{D})$ of \overline{D} to be

$$\widehat{\text{vol}}(\overline{D}) := \limsup_{n \rightarrow \infty} \frac{\hat{h}^0(X, n\overline{D})}{n^{d+1}/(d+1)!}$$

and

$$\widehat{\text{vol}}_\chi(\overline{D}) := \limsup_{n \rightarrow \infty} \frac{\hat{\chi}(X, n\overline{D})}{n^{d+1}/(d+1)!},$$

respectively, where $d = \dim X$. By Minkowski's theorem, $\widehat{\text{vol}}_\chi(\overline{D}) \leq \widehat{\text{vol}}(\overline{D})$. Let $\overline{L} = (L, h)$ be another adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Clearly, if $\overline{L} \leq \overline{D}$, then $\hat{h}^0(X, \overline{L}) \leq \hat{h}^0(X, \overline{D})$ and $\widehat{\text{vol}}(\overline{L}) \leq \widehat{\text{vol}}(\overline{D})$. Further, by (1.3.3.1), if $\overline{L} \leq \overline{D}$ and $L = D$, then $\hat{\chi}(X, \overline{L}) \leq \hat{\chi}(X, \overline{D})$ and $\widehat{\text{vol}}_\chi(\overline{L}) \leq \widehat{\text{vol}}_\chi(\overline{D})$. For the symbol \mathfrak{b} , $\widehat{\text{vol}}_\mathfrak{b}(\overline{D})$ stands for either $\widehat{\text{vol}}(\overline{D})$ or $\widehat{\text{vol}}_\chi(\overline{D})$, that is,

$$\widehat{\text{vol}}_\mathfrak{b}(\overline{D}) = \begin{cases} \widehat{\text{vol}}(\overline{D}) & \text{if } \mathfrak{b} \text{ is blank,} \\ \widehat{\text{vol}}_\chi(\overline{D}) & \text{if } \mathfrak{b} \text{ is } \chi. \end{cases}$$

4.4. Positivity of adelic arithmetic \mathbb{R} -Cartier divisors. Here let us introduce several kinds of the positivity of adelic arithmetic divisors.

Definition 4.4.1. Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X .

- **Big:** We say \overline{D} is *big* if $\widehat{\text{vol}}(\overline{D}) > 0$. According as [21], we can give an alternative definition, that is, for any adelic arithmetic \mathbb{R} -Cartier divisor \overline{L} of C^0 -type on X , $\hat{H}^0(X, n\overline{D} + \overline{L}) \neq \{0\}$ for some positive integer n . Actually two definitions are equivalent by the continuity of the volume function.

- **Pseudo-effective:** \overline{D} is said to be *pseudo-effective* if $\overline{D} + \overline{A}$ is big for any big adelic arithmetic \mathbb{R} -Cartier divisor \overline{A} of C^0 -type on X .

- **Relatively nef:** \overline{D} is said to be *relatively nef* if the following conditions are satisfied:

- (1) For $P \in M_K$, g_P is of $(C^0 \cap \text{PSH})$ -type.
- (2) The Green function g_∞ on $X(\mathbb{C})$ is of $(C^0 \cap \text{PSH})$ -type, that is, the first Chern current $c_1(D, g_\infty)$ is positive.

If $(\mathcal{X}_U, \mathcal{D}_U)$ is a defining model of \overline{D} , then \mathcal{D}_U is relatively nef with respect to $\mathcal{X}_U \rightarrow U$ by Proposition 2.1.7.

• **Nef:** We say \overline{D} is *nef* if \overline{D} is relatively nef and $\widehat{\deg}(\overline{D}|_x) \geq 0$ for all closed point x of X . For example, if $\phi \in \text{Rat}(X)^\times$, then the adelic arithmetic principal divisor $\widehat{(\phi)}$ of ϕ is nef.

Let us see the following proposition.

Proposition 4.4.2. *Let $\overline{D} = (D, \{g_P\}_{P \in M_K} \cup \{g_\infty\})$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Then we have the following:*

- (1) *Let U be a non-empty open set of $\text{Spec}(O_K)$ such that there is a defining model $(\mathcal{X}_U, \mathcal{D}_U)$ of \overline{D} over U . If \overline{D} is relatively nef, then there are sequences $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^\infty$ and $\{(\mathcal{X}'_n, \mathcal{D}'_n)\}_{n=1}^\infty$ with the following properties:*
 - (1.1) *For every $n \geq 1$, \mathcal{X}_n is a normal model of X over $\text{Spec}(O_K)$ such that $\mathcal{X}_n|_U = \mathcal{X}_U$.*
 - (1.2) *For every $n \geq 1$, \mathcal{D}_n and \mathcal{D}'_n are relatively nef \mathbb{R} -Cartier divisors on \mathcal{X}_n such that $\mathcal{D}_n|_U = \mathcal{D}'_n|_U = \mathcal{D}_U$.*
 - (1.3) *$(\mathcal{D}'_n, g_\infty)^a \leq \overline{D} \leq (\mathcal{D}_n, g_\infty)^a$ for all $n \geq 1$.*
 - (1.4) *If we set*

$$\phi_{n,P} := g_P - g_{((\mathcal{X}_n)_{(P)}, (\mathcal{D}_n)_{(P)})} \quad \text{and} \quad \phi'_{n,P} := g_P - g_{((\mathcal{X}'_n)_{(P)}, (\mathcal{D}'_n)_{(P)})},$$

then

$$\lim_{n \rightarrow \infty} \|\phi_{n,P}\|_{\text{sup}} = \lim_{n \rightarrow \infty} \|\phi'_{n,P}\|_{\text{sup}} = 0,$$

where $(\mathcal{X}_n)_{(P)}$ is the localization of $\mathcal{X}_n \rightarrow \text{Spec}(O_K)$ at P and $(\mathcal{D}_n)_{(P)}$ and $(\mathcal{D}'_n)_{(P)}$ are the restrictions of \mathcal{D}_n and \mathcal{D}'_n to $(\mathcal{X}_n)_{(P)}$, respectively.

- (2) *If \overline{D} is nef, then \overline{D} is pseudo-effective.*
- (3) *If D is big on X and \overline{D} is pseudo-effective, then $\overline{D} + (0, \epsilon[\infty])$ is big for any positive number ϵ .*

Proof. (1) By Proposition 2.1.7, \mathcal{D}_U is relatively nef with respect to $\mathcal{X}_U \rightarrow U$. Thus the assertion follows from Proposition 2.1.8.

(2) Let us choose a non-empty open set U of $\text{Spec}(O_K)$ such that \overline{D} has a defining model over U . Let \mathcal{X} be a normal model of X over $\text{Spec}(O_K)$ and let $\overline{\mathcal{A}}$ be an arithmetic Cartier divisor of C^∞ -type on \mathcal{X} such that

$$\overline{\mathcal{A}} - \left(\sum_{P \in M_K \setminus U} F_P, 0 \right)$$

is ample, where F_P is the fiber of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ over P . It is sufficient to show that $\overline{D} + \epsilon \overline{\mathcal{A}}$ is big for all $\epsilon \in \mathbb{R}_{>0}$. By (1), we can choose a normal model \mathcal{X}' of X over $\text{Spec}(O_K)$ and a relatively nef \mathbb{R} -Cartier divisor \mathcal{D} on \mathcal{X}' such that

$$(\mathcal{D}, g_\infty)^a - \left(0, \sum_{P \in M_K \setminus U} \epsilon[P] \right) \leq \overline{D} \leq (\mathcal{D}, g_\infty)^a.$$

We may assume that there is a birational morphism $\mu : \mathcal{X}' \rightarrow \mathcal{X}$. Then (\mathcal{D}, g_∞) is nef by Lemma 4.2.3 and

$$\left((\mathcal{D}, g_\infty) + \epsilon \mu^* \left(\overline{\mathcal{A}} - \left(\sum_{P \in M_K \setminus U} F_P, 0 \right) \right) \right)^a \leq \overline{D} + \epsilon \overline{\mathcal{A}}.$$

Note that $(\mathcal{D}, g_\infty) + \epsilon \mu^* \left(\overline{\mathcal{A}} - \left(\sum_{P \in M_K \setminus U} F_P, 0 \right) \right)$ is nef and big by [26, Proposition 6.2.2], as required.

(3) Let \bar{A} be a big adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Let us see the following claim:

Claim 4.4.2.1. *There are $m \in \mathbb{Z}_{>0}$, $\phi \in \text{Rat}(X)^\times$ and $\lambda \in \mathbb{R}$ such that*

$$m\bar{D} - \bar{A} + (\widehat{\phi}) + (0, \lambda[\infty]) \geq 0.$$

Proof. Since D is big on X , there are a positive integer m and a non-zero rational function ψ on X such that $m\bar{D} - \bar{A} + (\widehat{\psi})$ is effective. We set $(L, h) := m\bar{D} - \bar{A} + (\widehat{\psi})$. Let U be a non-empty open set of $\text{Spec}(O_K)$ such that \bar{L} has a defining model \mathcal{L}_U over U . As $L \geq 0$, shrinking U if necessarily, we may assume that \mathcal{L}_U is effective. In particular, $h_P \geq 0$ for $P \in M_K \cap U$. Thus there is $\lambda' \in \mathbb{R}$ such that

$$(L, h) \geq \left(0, \sum_{P \in M_K \setminus U} (-\lambda')[P] + (-\lambda')[\infty]\right).$$

We choose $N \in \mathbb{Z}_{>0}$ such that $-\lambda' + 2 \text{ord}_P(N) \#(O_K/P) \geq 0$ for all $P \in M_K \setminus U$. If we set $\Delta = m\bar{D} - \bar{A} + (\widehat{N\psi})$, then

$$\begin{aligned} \Delta &= (L, h) + (\widehat{N}) \\ &\geq (L, h) + \left(0, \sum_{P \in M_K \setminus U} 2 \text{ord}_P(N) \#(O_K/P)[P] - \log N^2[\infty]\right) \\ &\geq \left(0, \sum_{P \in M_K \setminus U} (-\lambda' + 2 \text{ord}_P(N) \#(O_K/P))[P] - (\lambda' + \log N^2)[\infty]\right) \\ &\geq (0, -(\lambda' + \log N^2)[\infty]), \end{aligned}$$

as required. \square

Let n be a positive integer such that $\lambda/(n+m) \leq \epsilon$. Then

$$\begin{aligned} \bar{D} + (1/n)(\bar{A} - (\widehat{\phi})) &\leq \bar{D} + (1/n)(m\bar{D} + (0, \lambda[\infty])) \\ &= ((n+m)/n)(\bar{D} + (0, \lambda/(n+m)[\infty])) \\ &\leq ((n+m)/n)(\bar{D} + (0, \epsilon[\infty])), \end{aligned}$$

so that we have the assertion. \square

In addition to the above positivity, an adelic arithmetic \mathbb{R} -Cartier divisor \bar{D} of C^0 -type on X is said to be *integrable* if there are relatively nef adelic arithmetic \mathbb{R} -Cartier divisors \bar{D}' and \bar{D}'' of C^0 -type on X such that $\bar{D} = \bar{D}' - \bar{D}''$. The set of all integrable adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X is denoted by $\widehat{\text{Div}}_{\text{int}}^a(X)_{\mathbb{R}}$. Note that $\widehat{\text{Div}}_{\text{int}}^a(X)_{\mathbb{R}}$ forms a subspace of $\widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$ over \mathbb{R} .

Remark 4.4.3. Let \mathcal{X} be a normal model of X over $\text{Spec}(O_K)$. We recall that an arithmetic \mathbb{R} -Cartier divisor $\bar{\mathcal{D}}$ of C^0 -type on \mathcal{X} is said to be *integrable* if there are relatively nef arithmetic \mathbb{R} -Cartier divisors $\bar{\mathcal{D}}'$ and $\bar{\mathcal{D}}''$ of C^0 -type on \mathcal{X} such that $\bar{\mathcal{D}} = \bar{\mathcal{D}}' - \bar{\mathcal{D}}''$ (cf. [27, Subsection 2.1]).

Finally let us introduce the relative nefness of a global adelic \mathbb{R} -Cartier divisor.

Definition 4.4.4. Let $\bar{D} = (D, \{g_P\}_{P \in M_K})$ be a global adelic \mathbb{R} -Cartier divisor of C^0 -type on X . We say \bar{D} is *relatively nef* if g_P is of $(C^0 \cap \text{PSH})$ -type for all $P \in M_K$.

4.5. Global intersection number. The purpose of this subsection is to construct the intersection pairing

$$\left(\widehat{\text{Div}}_{\text{int}}^a(X)_{\mathbb{R}}\right)^{d+1} \rightarrow \mathbb{R} \quad \left((\overline{D}_1, \dots, \overline{D}_{d+1}) \mapsto \widehat{\deg}(\overline{D} \cdots \overline{D}_{d+1})\right)$$

by using the local intersection number (cf. Subsection 2.4). For this, let us begin with the following lemma.

Lemma 4.5.1. *Let $\overline{D} = (D, g)$ be an integrable adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Then there are a normal model \mathcal{X} of X , an integrable arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on \mathcal{X} , a finite subset $\{P_1, \dots, P_r\}$ of M_K and integrable continuous functions ϕ_1, \dots, ϕ_r on $X_{P_1}^{\text{an}}, \dots, X_{P_r}^{\text{an}}$, respectively such that*

$$\overline{D} = \overline{\mathcal{D}}^a + \left(0, \sum_{i=1}^r \phi_i[P_i]\right).$$

Proof. By definition, there are relatively nef adelic arithmetic \mathbb{R} -Cartier divisors \overline{L}_1 and \overline{L}_2 of C^0 -type on X such that $\overline{D} = \overline{L}_1 - \overline{L}_2$, so that, by using Proposition 4.4.2, we can find a normal model \mathcal{X} of X , relatively nef arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ of C^0 -type on \mathcal{X} and a finite subset $\{P_1, \dots, P_r\}$ of M_K such that

$$\overline{L}_1 = \overline{\mathcal{L}}_1^a + \left(0, \sum_{i=1}^r \varphi_i[P_i]\right) \quad \text{and} \quad \overline{L}_2 = \overline{\mathcal{L}}_2^a + \left(0, \sum_{i=1}^r \psi_i[P_i]\right),$$

where φ_i and ψ_i are continuous functions on $X_{P_i}^{\text{an}}$. Note that φ_i and ψ_i are integrable. Thus, if we set $\overline{\mathcal{D}} = \overline{\mathcal{L}}_1 - \overline{\mathcal{L}}_2$ and $\phi_i = \varphi_i - \psi_i$, then we have the assertion. \square

Let $\overline{D}_1 = (D_1, g_1), \dots, \overline{D}_{d+1} = (D_{d+1}, g_{d+1})$ be integrable adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Then, by Lemma 4.5.1, there are a normal model \mathcal{X} of X , integrable arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}$ of C^0 -type on \mathcal{X} , and a finite subset S of M_K such that

$$\overline{D}_i = \overline{\mathcal{D}}_i^a + \left(0, \sum_{P \in S} \phi_{i,P}[P]\right),$$

where $\phi_{i,P}$'s are integrable continuous functions X_P^{an} . We would like to define the intersection number $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1})$ to be

$$\begin{aligned} \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1}) &:= \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1}) \\ &+ \sum_{P \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/P) \widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(P)} \right), \end{aligned}$$

where $\widehat{\deg}_P$ is the local intersection number at P (cf. Subsection 2.4) and $(\mathcal{D}_j)_{(P)}$ means the restriction of \mathcal{D}_j to the localization of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ at P . For this purpose, we need to see that the above formula does not depend on the choice of \mathcal{X} , $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}$ and S . We denote the right hand side of the above by $\Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}, S)$. Let \mathcal{X}' , $\overline{\mathcal{D}}'_1, \dots, \overline{\mathcal{D}}'_{d+1}$ and S' be another choice. In order to check

$$\Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}, S) = \Delta(\mathcal{X}', \overline{\mathcal{D}}'_1, \dots, \overline{\mathcal{D}}'_{d+1}, S'),$$

we may assume that $\mathcal{X}' = \mathcal{X}$ and $S' = S$. Note that there are vertical \mathbb{R} -Cartier divisors $\mathcal{E}_1, \dots, \mathcal{E}_{d+1}$ on \mathcal{X} such that $\text{Supp}_{\mathbb{R}}(\mathcal{E}_1), \dots, \text{Supp}_{\mathbb{R}}(\mathcal{E}_{d+1}) \subseteq \sum_{P \in S} F_P$ and $\mathcal{D}'_i = \mathcal{D}_i + \mathcal{E}_i$ for $i = 1, \dots, d+1$, where F_P is the fiber of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ over P . Thus it is sufficient to show that

$$\Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}, S) = \Delta(\mathcal{X}, \overline{\mathcal{D}}_1 + (\mathcal{E}_1, 0), \dots, \overline{\mathcal{D}}_{d+1} + (\mathcal{E}_{d+1}, 0), S)$$

for all vertical \mathbb{R} -Cartier divisors $\mathcal{E}_1, \dots, \mathcal{E}_{d+1}$ on \mathcal{X} with

$$\text{Supp}_{\mathbb{R}}(\mathcal{E}_1), \dots, \text{Supp}_{\mathbb{R}}(\mathcal{E}_{d+1}) \subseteq \sum_{P \in S} F_P.$$

If we can show

$$(4.5.2) \quad \Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_l, \dots, \overline{\mathcal{D}}_{d+1}, S) = \Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_l + (\mathcal{E}, 0), \dots, \overline{\mathcal{D}}_{d+1}, S)$$

for a vertical \mathbb{R} -Cartier divisor \mathcal{E} on \mathcal{X} with $\text{Supp}_{\mathbb{R}}(\mathcal{E}) \subseteq \sum_{P \in S} F_P$, then

$$\begin{aligned} \Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_{d+1}, S) &= \Delta(\mathcal{X}, \overline{\mathcal{D}}_1 + (\mathcal{E}_1, 0), \overline{\mathcal{D}}_2, \dots, \overline{\mathcal{D}}_{d+1}, S) \\ &= \Delta(\mathcal{X}, \overline{\mathcal{D}}_1 + (\mathcal{E}_1, 0), \overline{\mathcal{D}}_2 + (\mathcal{E}_2, 0), \overline{\mathcal{D}}_3, \dots, \overline{\mathcal{D}}_{d+1}, S) \\ &= \dots = \Delta(\mathcal{X}, \overline{\mathcal{D}}_1 + (\mathcal{E}_1, 0), \dots, \overline{\mathcal{D}}_{d+1} + (\mathcal{E}_{d+1}, 0), S). \end{aligned}$$

Therefore, it suffices to check (4.5.2). We set $e_P = g_{(\mathcal{X}(P), \mathcal{E}(P))}$. Then

$$\overline{D}_l = (\overline{\mathcal{D}}_l + (\mathcal{E}, 0))^a + \left(0, \sum_{P \in S} (\phi_{l,P} - e_P)[P] \right),$$

so that

$$\begin{aligned} \Delta(\mathcal{X}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_l + (\mathcal{E}, 0), \dots, \overline{\mathcal{D}}_{d+1}, S) &= \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots (\overline{\mathcal{D}}_l + (\mathcal{E}, 0)) \cdots \overline{\mathcal{D}}_{d+1}) + \\ &\quad \sum_{P \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/P) \widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P} - \delta_{il} e_P) \cdot \prod_{j \notin I} (\mathcal{D}_j + \delta_{jl} \mathcal{E})_{(P)} \right). \end{aligned}$$

Note that

$$\begin{aligned} \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots (\overline{\mathcal{D}}_l + (\mathcal{E}, 0)) \cdots \overline{\mathcal{D}}_{d+1}) &= \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1}) \\ &\quad + \sum_{P \in S} \log \#(O_K/P) \widehat{\deg}_P((\mathcal{D}_1)_{(P)} \cdots (\mathcal{D}_{l-1})_{(P)} \cdot (\mathcal{D}_{l+1})_{(P)} \cdots (\mathcal{D}_{d+1})_{(P)} \cdot \mathcal{E}_{(P)}). \end{aligned}$$

Moreover,

$$\widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P} - \delta_{il} e_P) \cdot \prod_{j \notin I} (\mathcal{D}_j + \delta_{jl} \mathcal{E})_{(P)} \right) = \begin{cases} \widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(P)} \right) \\ \quad - \widehat{\deg}_P \left(\prod_{i \in I \setminus \{l\}} (0, \phi_{i,P}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(P)} \cdot e_P \right) & \text{if } l \in I, \\ \widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(P)} \right) \\ \quad + \widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P}) \cdot \prod_{j \notin I \cup \{l\}} (\mathcal{D}_j)_{(P)} \cdot e_P \right) & \text{if } l \notin I, \end{cases}$$

and hence

$$\begin{aligned} & \sum_{P \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/P) \widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P} - \delta_{il} e_P) \cdot \prod_{j \notin I} (\mathcal{D}_j + \delta_{jl} \mathcal{E})_{(P)} \right) \\ &= \sum_{P \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/P) \widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(P)} \right) \\ & \quad - \sum_{P \in S} \log \#(O_K/P) \widehat{\deg}_P ((\mathcal{D}_1)_{(P)} \cdots (\mathcal{D}_{l-1})_{(P)} \cdot (\mathcal{D}_{l+1})_{(P)} \cdots (\mathcal{D}_{d+1})_{(P)} \cdot e_P). \end{aligned}$$

Therefore, we have (4.5.2). By our construction, it is easy to see that the map

$$(\overline{D}_1, \dots, \overline{D}_{d+1}) \mapsto \widehat{\deg}(\overline{D} \cdots \overline{D}_{d+1})$$

is multi-linear and symmetric (cf. (2.4.6), (2.4.7) and Proposition 2.4.8).

Let $\overline{D}_1, \dots, \overline{D}_{d+1}, \overline{D}'_1, \dots, \overline{D}'_{d+1}$ be integrable adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Let T be a finite set of M_K and let $\varphi_{1,P}, \dots, \varphi_{d,P}$ be integrable continuous functions on X_P^{an} for $P \in T$. By using Lemma 1.3.5, we can see that if

$$\overline{D}'_i = \overline{D}_i + \left(0, \sum_{P \in T} \varphi_{i,P} [P] \right)$$

for $i = 1, \dots, d+1$, then

$$(4.5.3) \quad \widehat{\deg}(\overline{D}'_1 \cdots \overline{D}'_{d+1}) = \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1}) + \sum_{P \in T} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/P) \widehat{\deg}_P \left(\prod_{i \in I} (0, \varphi_{i,P}) \cdot \prod_{j \notin I} (D_j, (g_j)_P) \right),$$

where $\overline{D}_j = (D_j, \sum_P (g_j)_P [P] + (g_j)_\infty [\infty])$ for $j = 1, \dots, d+1$.

Proposition 4.5.4. *Let $\overline{D}_1, \dots, \overline{D}_d, \overline{D}_{d+1}$ be integrable adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Then we have the following:*

- (1) *For $\phi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \widehat{(\phi)}) = 0$.*
- (2) *If $\overline{D}_1, \dots, \overline{D}_d$ are nef and \overline{D}_{d+1} is pseudo-effective, then*

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}) \geq 0.$$

Proof. (1) Let us begin with the following claim:

Claim 4.5.4.1. *Let \mathcal{X} be a normal model of X and let $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ be integrable arithmetic \mathbb{R} -Cartier divisors of C^0 -type on \mathcal{X} . Then $\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \widehat{(\phi)}) = 0$.*

Proof. **Step 1** (the case where $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are of C^∞ -type) : If $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are arithmetic Cartier divisors of C^∞ -type on \mathcal{X} and $\phi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, then the assertion is well-known. On the other hand, by [26, Proposition 2.4.2], we can find arithmetic Cartier divisors $\overline{\mathcal{E}}_1, \dots, \overline{\mathcal{E}}_r$ of C^∞ -type on \mathcal{X} , $\phi_1, \dots, \phi_l \in \text{Rat}(X)_{\mathbb{R}}^{\times}$, $a_{ij} \in \mathbb{R}$ ($i = 1, \dots, d$, $j = 1, \dots, r$) and $b_1, \dots, b_l \in \mathbb{R}$ such that $\overline{\mathcal{D}}_i = \sum_{j=1}^r a_{ij} \overline{\mathcal{E}}_j$ and $\phi = \phi_1^{b_1} \cdots \phi_l^{b_l}$. Thus, using the linearity of the intersection pairing, we have the assertion.

Step 2 (the case where $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are relatively nef) : Let $\overline{\mathcal{A}}$ be an ample arithmetic Cartier divisor of C^∞ -type on \mathcal{X} . As

$$\lim_{n \rightarrow \infty} \widehat{\deg}((\overline{\mathcal{D}}_1 + (1/n)\overline{\mathcal{A}}) \cdots (\overline{\mathcal{D}}_d + (1/n)\overline{\mathcal{A}}) \cdot \widehat{(\phi)}) = \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \widehat{(\phi)}),$$

we may assume that $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ is ample on \mathcal{X} . Then, by [26, Theorem 4.6], there are sequences $\{f_{1,n}\}_{n=1}^\infty, \dots, \{f_{d,n}\}_{n=1}^\infty$ of F_∞ -invariant continuous functions on $X(\mathbb{C})$ such that

$$\lim_{n \rightarrow \infty} \|f_{i,n}\|_{\sup} = 0$$

and $\overline{\mathcal{D}}_i + (0, f_{i,n})$ is relatively nef and of C^∞ -type for $i = 1, \dots, d$ and $n \geq 1$. Therefore, by using [27, Lemma 1.2.1] together with Step 1, we have

$$\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \widehat{(\phi)}) = \lim_{n \rightarrow \infty} \widehat{\deg}((\overline{\mathcal{D}}_1 + (0, f_{1,n})) \cdots (\overline{\mathcal{D}}_d + (0, f_{d,n})) \cdot \widehat{(\phi)}) = 0.$$

Step 3 (general case) : Since $\overline{\mathcal{D}}_i$ is integrable, there are relatively nef arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{L}}_i$ and $\overline{\mathcal{M}}_i$ of C^0 -type on \mathcal{X} such that $\overline{\mathcal{D}}_i = \overline{\mathcal{L}}_i - \overline{\mathcal{M}}_i$. Thus the assertion follows from Step 2. \square

Let us start the proof of (1). By Lemma 4.5.1, we can find a normal model \mathcal{X} of X , integrable arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ of C^0 -type on \mathcal{X} , and a finite subset S of M_K such that

$$\overline{D}_i = \overline{\mathcal{D}}_i^a + \left(0, \sum_{P \in S} \phi_{i,P}[P]\right),$$

where $\phi_{i,P}$'s are integrable continuous functions X_p^{an} . Then, by (4.5.3),

$$\begin{aligned} \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \widehat{(\phi)}) &= \widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \widehat{(\phi)}) \\ &+ \sum_{P \in S} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \log \#(O_K/P) \widehat{\deg}_P \left(\prod_{i \in I} (0, \phi_{i,P}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(P)} \cdot (\phi)_{(P)} \right). \end{aligned}$$

Therefore, (1) follows from Claim 4.5.4.1 and (1) in Proposition 2.4.10.

(2) First let us see the following claim:

Claim 4.5.4.2. *Let \mathcal{X} be a normal model of X and $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d, \overline{\mathcal{D}}_{d+1}$ be integrable arithmetic \mathbb{R} -Cartier divisors of C^0 -type on \mathcal{X} .*

(a) *If $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are nef and $\overline{\mathcal{D}}_{d+1}$ is effective, then*

$$\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_d \cdot \overline{\mathcal{D}}_{d+1}) \geq 0.$$

(b) *Let \overline{E} be an effective and integrable adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . If $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are nef, then*

$$\widehat{\deg}(\overline{\mathcal{D}}_1^a \cdots \overline{\mathcal{D}}_d^a \cdot \overline{E}) \geq 0.$$

Proof. (a) Let $\overline{\mathcal{A}}$ be an ample arithmetic Cartier divisor of C^∞ -type on \mathcal{X} . It is sufficient to show that

$$\widehat{\deg}((\overline{\mathcal{D}}_1 + \epsilon \overline{\mathcal{A}}) \cdots (\overline{\mathcal{D}}_d + \epsilon \overline{\mathcal{A}}) \cdot \overline{\mathcal{D}}_{d+1}) \geq 0$$

for $\epsilon > 0$. First we assume that $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_d$ are of C^∞ -type. Then, by [26, Proposition 6.2.2], $\overline{\mathcal{D}}_i + \epsilon \overline{\mathcal{A}}$ is ample for every i , that is, there are ample arithmetic Cartier divisors $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_r$ of C^∞ -type on \mathcal{X} such that $\overline{\mathcal{D}}_i + \epsilon \overline{\mathcal{A}} = \sum_{j=1}^r a_{ij} \overline{\mathcal{A}}_j$ for some $a_{ij} \in \mathbb{R}_{\geq 0}$. On the other hand, by [26, Proposition 2.4.2], there are effective arithmetic Cartier divisors $\overline{\mathcal{E}}_1, \dots, \overline{\mathcal{E}}_l$ of C^∞ -type on \mathcal{X} such that $\overline{\mathcal{D}}_{d+1} = b_1 \overline{\mathcal{E}}_1 + \cdots + b_l \overline{\mathcal{E}}_l$ for some $b_1, \dots, b_l \in \mathbb{R}_{\geq 0}$, and hence the assertion follows from [21, Proposition 2.3].

In general, as before, by [26, Theorem 4.6], there are sequences

$$\{f_{1,n}\}_{n=1}^\infty, \dots, \{f_{d,n}\}_{n=1}^\infty$$

of F_∞ -invariant non-negative continuous functions on $X(\mathbb{C})$ such that

$$\lim_{n \rightarrow \infty} \|f_{i,n}\|_{\sup} = 0$$

and $\overline{\mathcal{D}}_i + \epsilon \overline{\mathcal{A}} + (0, f_{i,n})$ is nef and of C^∞ -type for $i = 1, \dots, d$ and $n \geq 1$. Therefore, by [27, Lemma 1.2.1], we have

$$\begin{aligned} & \widehat{\deg}((\overline{\mathcal{D}}_1 + \epsilon \overline{\mathcal{A}}) \cdots (\overline{\mathcal{D}}_d + \epsilon \overline{\mathcal{A}}) \cdot \overline{\mathcal{D}}_{d+1}) \\ &= \lim_{n \rightarrow \infty} \widehat{\deg}((\overline{\mathcal{D}}_1 + \epsilon \overline{\mathcal{A}} + (0, f_{1,n})) \cdots (\overline{\mathcal{D}}_d + \epsilon \overline{\mathcal{A}} + (0, f_{d,n})) \cdot \overline{\mathcal{D}}_{d+1}) \geq 0, \end{aligned}$$

as required.

(b) By Theorem 4.1.3, there is a finite subset S of M_K with the following property: for any $n \in \mathbb{Z}_{>0}$, there are a normal model \mathcal{X}_n of X together with a birational morphism $\mu_n: \mathcal{X}_n \rightarrow \mathcal{X}$, and an integrable arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{E}}_n$ of C^0 -type on \mathcal{X}_n such that

$$\overline{E} \leq \overline{\mathcal{E}}_n^a \leq \overline{E} + \left(0, \sum_{P \in S} (1/n)[P]\right).$$

Then $\overline{\mathcal{E}}_n$ is effective by (2) in Proposition 4.1.2 and, if we set

$$\overline{\mathcal{E}}_n^a = \overline{E} + \left(0, \sum_{P \in S} \psi_{n,P}[P]\right),$$

then $\psi_{n,P}$ is a continuous function on X_P^{an} with $0 \leq \psi_{n,P} \leq 1/n$ for every n and $P \in S$. Therefore, by (a)

$$\begin{aligned} \widehat{\deg}(\overline{\mathcal{D}}_1^a \cdots \overline{\mathcal{D}}_d^a \cdot \overline{E}) &= \widehat{\deg}(\mu_n^*(\overline{\mathcal{D}}_1) \cdots \mu_n^*(\overline{\mathcal{D}}_d) \cdot \overline{\mathcal{E}}_n) \\ &\quad - \sum_{P \in S} \log \#(O_K/P) \widehat{\deg}_P(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi_{n,P}) \\ &\geq - \sum_{P \in S} \log \#(O_K/P) \widehat{\deg}_P(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi_{n,P}) \end{aligned}$$

On the other hand, by (3) in Proposition 2.4.10,

$$\left| \widehat{\deg}_P(\mathcal{L}_1 \cdots \mathcal{L}_d; \psi_{n,P}) \right| \leq \frac{1}{-2n \log v(\varpi)} \deg((\mathcal{L}_1 \cap X) \cdots (\mathcal{L}_d \cap X)).$$

Thus we have the assertion of (b). \square

Claim 4.5.4.3. *If $\overline{D}_1, \dots, \overline{D}_d$ are nef and \overline{D}_{d+1} is effective, then*

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}) \geq 0.$$

Proof. By (1) in Proposition 4.4.2, as before, we can find a finite subset T of M_K with the following property: for any $n \in \mathbb{Z}_{>0}$, there are a normal model \mathcal{X}_n and relatively nef arithmetic \mathbb{R} -Cartier divisors

$$\overline{\mathcal{D}}_{1,n}, \dots, \overline{\mathcal{D}}_{d,n}$$

of C^0 -type on \mathcal{X}_n such that

$$\overline{D}_i \leq \overline{\mathcal{D}}_{i,n}^a \leq \overline{D}_i + \left(0, \sum_{P \in T} (1/n)[P]\right)$$

for $i = 1, \dots, d$. Note that $\overline{\mathcal{D}}_{i,n}$ is nef for every i and n by Lemma 4.2.3, and if we set

$$\overline{\mathcal{D}}_{i,n}^a = \overline{D}_i + \left(0, \sum_{P \in T} \varphi_{i,n,P}[P]\right),$$

then $\varphi_{i,n,P}$ is a continuous function on X_P^{an} with $0 \leq \varphi_{i,n,P} \leq 1/n$ for every n and $P \in T$. Then, by (4.5.3),

$$\begin{aligned} \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}) &= \widehat{\deg}(\overline{\mathcal{D}}_{1,n}^a \cdots \overline{\mathcal{D}}_{d,n}^a \cdot \overline{D}_{d+1}) + \sum_{P \in T} \sum_{\substack{I \subseteq \{1, \dots, d+1\} \\ I \neq \emptyset}} \\ &\quad \log \#(O_K/P) \widehat{\deg}_P \left(\prod_{i \in I} (0, -\varphi_{i,n,P}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(P)} \cdot (D_{d+1}, (g_{d+1})_P) \right), \end{aligned}$$

where $\overline{D}_{d+1} = (D_{d+1}, \sum_P (g_{d+1})_P[P] + (g_{d+1})_\infty[\infty])$. By using (3) in Proposition 2.4.10, it is easy to see that

$$\lim_{n \rightarrow \infty} \widehat{\deg}_P \left(\prod_{i \in I} (0, -\varphi_{i,n,P}) \cdot \prod_{j \notin I} (\mathcal{D}_j)_{(P)} \cdot (D_{d+1}, (g_{d+1})_P) \right) = 0$$

for all $P \in T$ and $I \subseteq \{1, \dots, d\}$ with $I \neq \emptyset$ (cf. the proof of (2) in Claim 4.5.4.2). Therefore, the assertion follows from (2) in Claim 4.5.4.2. \square

Let \overline{B} be a nef and big adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . For $\epsilon > 0$, as $\overline{D}_{d+1} + \epsilon \overline{B}$ is big, there are $n \in \mathbb{Z}_{>0}$ and $\psi \in \text{Rat}(X)^\times$ such that

$$\overline{D}_{d+1} + \epsilon \overline{B} + (1/n)(\widehat{\psi}) \geq 0,$$

so that, by using (1) and Claim 4.5.4.3,

$$\begin{aligned} \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{D}_{d+1}) + \epsilon \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{B}) \\ = \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d \cdot (\overline{D}_{d+1} + \epsilon \overline{B} + (1/n)(\widehat{\psi}))) \geq 0. \end{aligned}$$

Thus the assertion follows. \square

5. CONTINUITY OF THE VOLUME FUNCTION

The purpose of this section is to consider the continuity of the volume function and its applications. The continuity of the volume function is a very fundamental result in the theory of birational Arakelov geometry. It actually has a lot of applications by using perturbation methods. The generalized Hodge index theorem is one of significant examples, which is a generalization of results due to Faltings-Gillet-Soulé-Zhang (cf. [9], [10] and [33]).

Throughout this section, let K be a number field and let X be a d -dimensional, projective, smooth and geometrically integral variety over K .

5.1. Basic properties of the volume. In this subsection, we investigate several basic properties of the volume function. First of all, let us begin with the finiteness of $\widehat{\text{vol}}$, the limit theorem and the positive homogeneity of $\widehat{\text{vol}}$.

Theorem 5.1.1. (1) (Finiteness) $\widehat{\text{vol}}(\overline{D}) < \infty$.

(2) (Limit theorem) *The “lim sup” in the definition of $\widehat{\text{vol}}$ (cf. Definition 4.3.2) can be replaced by “lim”, that is,*

$$\widehat{\text{vol}}(\overline{D}) = \lim_{t \rightarrow \infty} \frac{\hat{h}^0(X, t\overline{D})}{t^{d+1}/(d+1)!},$$

where t is a positive real number.

(3) (Positive homogeneity) $\widehat{\text{vol}}(a\overline{D}) = a^{d+1}\widehat{\text{vol}}(\overline{D})$ for $a \in \mathbb{R}_{\geq 0}$.

Proof. (1) is obvious because we can find a normal model \mathcal{X} of X over $\text{Spec}(O_K)$ and an arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on \mathcal{X} such that $\overline{D} \leq \overline{\mathcal{D}}^a$.

(2) Let $(\mathcal{X}_U, \mathcal{D}_U)$ be a defining model of \overline{D} over a non-empty open set U of $\text{Spec}(O_K)$. By Theorem 4.1.3, for any $\epsilon > 0$, there is a normal model \mathcal{X}_ϵ of X and an arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}}_\epsilon$ of C^0 -type on \mathcal{X}_ϵ such that $\mathcal{X}_\epsilon|_U = \mathcal{X}_U$, $\mathcal{D}_\epsilon|_U = \mathcal{D}_U$ and

$$\overline{\mathcal{D}}_\epsilon^a \leq \overline{D} \leq \overline{\mathcal{D}}_\epsilon^a + \left(0, \sum_{P \in S} 2\epsilon \log \#(O_K/P)[P]\right).$$

where $S = M_K \setminus U$. Note that the fiber F_P of $\mathcal{X}_\epsilon \rightarrow \text{Spec}(O_K)$ over P yields a constant function $2 \log \#(O_K/P)$ on X_P^{an} . Thus the above inequalities mean that

$$\overline{\mathcal{D}}_\epsilon^a \leq \overline{D} \leq \left(\overline{\mathcal{D}}_\epsilon + \epsilon \left(\sum_{P \in S} F_P, 0\right)\right)^a.$$

As $\widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon)$ and $\widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon + \epsilon(\sum_p F_p, 0))$ can be expressed by “lim” (cf. [23, Theorem 5.1] and [26, Theorem 5.2.2]), if we set

$$\Delta = \limsup_{t \rightarrow \infty} \frac{\hat{h}^0(X, t\overline{D})}{t^{d+1}/(d+1)!} - \liminf_{t \rightarrow \infty} \frac{\hat{h}^0(X, t\overline{D})}{t^{d+1}/(d+1)!},$$

then

$$0 \leq \Delta \leq \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + \epsilon\left(\sum_{P \in S} F_P, 0\right)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon).$$

Let us choose a positive number N such that $N \in P$ for all $P \in S$. Then $\sum_{P \in S} F_P \leq (N)$. Thus, by using [23, Proposition 4.6] and [26, Theorem 5.2.2],

$$\begin{aligned} \Delta &\leq \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + \epsilon\left(\sum_p F_p, 0\right)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &\leq \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + \epsilon((N), 0)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &= \epsilon^{d+1} \widehat{\text{vol}}\left((1/\epsilon)\overline{\mathcal{D}}_\epsilon + ((N), 0)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &= \epsilon^{d+1} \widehat{\text{vol}}\left((1/\epsilon)\overline{\mathcal{D}}_\epsilon + ((N), 0) - (\widehat{N})\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &= \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + \epsilon((N), 0) - \epsilon(\widehat{N})\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &= \widehat{\text{vol}}\left(\overline{\mathcal{D}}_\epsilon + (0, 2\epsilon \log N)\right) - \widehat{\text{vol}}(\overline{\mathcal{D}}_\epsilon) \\ &\leq \epsilon \log(N)(d+1)[K:\mathbb{Q}] \text{vol}(X, D). \end{aligned}$$

Note that $\log(N)(d+1)[K:\mathbb{Q}] \text{vol}(X, D)$ does not depend on ϵ , so that $\Delta = 0$.

(3) If $a = 0$, then the assertion is obvious. Otherwise, by using (2),

$$\begin{aligned} \widehat{\text{vol}}(a\overline{D}) &= \lim_{t \rightarrow \infty} \frac{\hat{h}^0(X, ta\overline{D})}{t^{d+1}/(d+1)!} = a^{d+1} \lim_{t \rightarrow \infty} \frac{\hat{h}^0(X, ta\overline{D})}{(ta)^{d+1}/(d+1)!} \\ &= a^{d+1} \widehat{\text{vol}}(\overline{D}), \end{aligned}$$

as desired. \square

Next let us consider the following estimate for the proof of the continuity of $\widehat{\text{vol}}$ and $\widehat{\text{vol}}_\chi$.

Proposition 5.1.2. *Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor on X . Let $\varphi_1 \in C^0(X_{P_1}^{\text{an}}), \dots, \varphi_r \in C^0(X_{P_r}^{\text{an}}), \varphi_\infty \in C_{F_\infty}^0(X(\mathbb{C}))$, where $P_1, \dots, P_r \in M_K$. Then*

$$\begin{aligned} \left| \widehat{\text{vol}}_{\mathfrak{f}}\left(\overline{D} + \left(0, \sum_{i=1}^r \varphi_i[P_i] + \varphi_\infty[\infty]\right)\right) - \widehat{\text{vol}}_{\mathfrak{f}}(\overline{D}) \right| \\ \leq \frac{(d+1)[K:\mathbb{Q}] \text{vol}(X, D)}{2} \left(\sum_{i=1}^r \|\varphi_i\|_{\text{sup}} + \|\varphi_\infty\|_{\text{sup}} \right), \end{aligned}$$

where $\widehat{\text{vol}}_{\mathfrak{f}}$ is either $\widehat{\text{vol}}$ or $\widehat{\text{vol}}_\chi$ (see Definition 4.3.2).

Let us begin with the following lemma:

Lemma 5.1.3. *Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X and $P \in M_K$. Then we have the following:*

$$(1) \quad \widehat{\text{vol}}_{\mathfrak{f}}(\overline{D} + (0, a[P])) \leq \widehat{\text{vol}}_{\mathfrak{f}}(\overline{D}) + \frac{(d+1)[K:\mathbb{Q}] \text{vol}(X, D)}{2[K_P:\mathbb{Q}_p]} a \text{ for } a \in \mathbb{R}_{\geq 0}, \text{ where } p \text{ is the prime number with } p\mathbb{Z} = \mathbb{Z} \cap P.$$

$$(2) \widehat{\text{vol}}_{\mathbb{Q}}(\overline{D} + (0, a[\infty])) \leq \widehat{\text{vol}}_{\mathbb{Q}}(\overline{D}) + \frac{(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2}a \text{ for } a \in \mathbb{R}_{\geq 0}.$$

Proof. (1) For each $n \in \mathbb{Z}_{>0}$, let a_n be the round down of $\frac{na}{2\text{ord}_p(p)\log\#(O_K/P)}$, that is,

$$a_n = \left\lfloor \frac{na}{2\text{ord}_p(p)\log\#(O_K/P)} \right\rfloor.$$

Then we have the following claim:

Claim 5.1.3.1. $p^{a_n}\hat{H}^0(X, n(\overline{D}^\tau + (0, a[P]))) \subseteq \hat{H}^0(X, n\overline{D}^\tau).$

Proof. Let $\phi \in \hat{H}^0(X, n(\overline{D}^\tau + (0, a[P])))$. Let $Q \in \text{Spec}(O_K)$. If $Q \neq P$, then

$$\|p^{a_n}\phi\|_{ng_Q} = v_Q(p)^{a_n}\|\phi\|_{ng_Q} \leq 1.$$

Otherwise (i.e. $Q = P$),

$$\begin{aligned} \|p^{a_n}\phi\|_{ng_P} &= v_P(p)^{a_n}\|\phi\|_{ng_P} = \exp\left(-a_n\text{ord}_P(p)\log\#(O_K/P) + \frac{na}{2}\right)\|\phi\|_{n(g_P+a)} \\ &\leq \exp\left(\frac{-na}{2\text{ord}_P(p)\log\#(O_K/P)}\text{ord}_P(p)\log\#(O_K/P) + \frac{na}{2}\right)\|\phi\|_{n(g_P+a)} \\ &= \|\phi\|_{n(g_P+a)} \leq 1, \end{aligned}$$

as required. □

We set

$$\begin{cases} Q_n = \text{Coker}\left(\hat{H}^0(X, n\overline{D}^\tau) \rightarrow \hat{H}^0(X, n(\overline{D}^\tau + (0, a[P])))\right), \\ r_n = \dim_{\mathbb{Q}} H^0(X, nD) = [K:\mathbb{Q}]\dim_K H^0(X, nD). \end{cases}$$

Then, by (1.3.3.4) and (1.3.3.5),

$$\begin{cases} \hat{h}^0(X, n(\overline{D} + (0, a[P]))) \leq \hat{h}^0(X, n\overline{D}) + \log\#(Q_n) + \log(6)r_n, \\ \hat{\chi}(X, n(\overline{D} + (0, a[P]))) = \hat{\chi}(X, n\overline{D}) + \log\#(Q_n). \end{cases}$$

By the above claim,

$$\begin{aligned} \log\#(Q_n) &\leq \log\#\left(\hat{H}^0(X, n(\overline{D}^\tau + (0, a[P]))) / p^{a_n}\hat{H}^0(X, n(\overline{D}^\tau + (0, a[P])))\right) \\ &= a_n r_n \log(p). \end{aligned}$$

Therefore, (1) follows because

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n r_n \log(p)}{n^{d+1}/(d+1)!} &= \frac{a(d+1)[K:\mathbb{Q}]\log(p)}{2\text{ord}_P(p)\log\#(O_K/P)} \lim_{n \rightarrow \infty} \frac{\dim_K H^0(X, nD)}{n^d/d!} \\ &= \frac{a(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2\text{ord}_P(p)[O_K/P:\mathbb{Z}/p\mathbb{Z}]} = \frac{a(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2[K_P:\mathbb{Q}_p]}. \end{aligned}$$

(2) By (1.3.3.2) and (1.3.3.3),

$$\begin{cases} \hat{h}^0(X, n(\overline{D} + (0, a[\infty]))) \leq \hat{h}^0(X, n\overline{D}) + (na/2)r_n + \log(3)r_n, \\ \hat{\chi}(X, n(\overline{D} + (0, a[\infty]))) = \hat{\chi}(X, n\overline{D}) + (na/2)r_n. \end{cases}$$

Therefore, (2) follows. □

Proof of Proposition 5.1.2. Let us start the proof of Proposition 5.1.2. First we check the following special cases:

Claim 5.1.4.1. *The following inequalities hold:*

- (1) $\left| \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} + (0, \varphi_1[P_1])) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) \right| \leq \frac{(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2} \|\varphi_1\|_{\text{sup}}.$
- (2) $\left| \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} + (0, \varphi_{\infty}[\infty])) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) \right| \leq \frac{(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2} \|\varphi_{\infty}\|_{\text{sup}}.$

Proof. (1) By using (1) in Lemma 5.1.3,

$$\begin{aligned} \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} + (0, \varphi_1[P_1])) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) &\leq \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} + (0, \|\varphi_1\|_{\text{sup}}[P_1])) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) \\ &\leq \frac{(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2} \|\varphi_1\|_{\text{sup}}. \end{aligned}$$

Applying (1) in Lemma 5.1.3 to the case where \overline{D} is $\overline{D} - (0, a[P_1])$, we have

$$\widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) \leq \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} - (0, a[P_1])) + (a/2)(d+1)[K:\mathbb{Q}]\text{vol}(X, D).$$

Therefore,

$$\begin{aligned} \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} + (0, \varphi_1[P_1])) &\leq \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D} - (0, \|\varphi_1\|_{\text{sup}}[P_1])) \\ &\leq \frac{(d+1)[K:\mathbb{Q}]\text{vol}(X, D)}{2} \|\varphi_1\|_{\text{sup}}. \end{aligned}$$

Thus we have (1).

(2) can be shown in the same way as (1) by using (2) in Lemma 5.1.3. \square

In general, we set

$$\overline{D}_j = \begin{cases} \overline{D} + (0, \varphi_{\infty}[\infty]) & \text{if } j = 0, \\ \overline{D} + \left(0, \sum_{i=1}^j \varphi_i[P_i] + \varphi_{\infty}[\infty]\right) & \text{if } j \geq 1. \end{cases}$$

Then, as

$$\begin{aligned} \left| \widehat{\text{vol}}_{\mathfrak{q}}\left(\overline{D} + \left(0, \sum_{i=1}^r \varphi_i[P_i] + \varphi_{\infty}[\infty]\right)\right) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) \right| \\ \leq \sum_{j=1}^r \left| \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}_j) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}_{j-1}) \right| + \left| \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}_0) - \widehat{\text{vol}}_{\mathfrak{q}}(\overline{D}) \right|, \end{aligned}$$

using the previous claim, we have the assertion. \square

As a consequence of the above estimate, we have the following proposition, so that we can deduce Fujita's approximation theorem for adelic arithmetic \mathbb{R} -Cartier divisors (cf. Theorem 5.1.6).

Proposition 5.1.5. *Let \overline{D} be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Then, for a positive number ϵ , there are a normal model of \mathcal{X} over $\text{Spec}(O_K)$ and arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}'$ of C^0 -type on \mathcal{X} such that*

$$\overline{\mathcal{D}}^a \leq \overline{D} \leq \overline{\mathcal{D}}'^a, \quad 0 \leq \widehat{\text{vol}}(\overline{D}) - \widehat{\text{vol}}(\overline{\mathcal{D}}) \leq \epsilon \quad \text{and} \quad 0 \leq \widehat{\text{vol}}(\overline{\mathcal{D}}') - \widehat{\text{vol}}(\overline{D}) \leq \epsilon.$$

Proof. Let U be a non-empty open set of $\text{Spec}(O_K)$ such that \overline{D} has defining models \mathcal{D}_U over U . We set $S = M_K \setminus U$. We choose a positive number ϵ' such that

$$\epsilon'(d+1)[K:\mathbb{Q}]\widehat{\text{vol}}(X, D)\#(S) \leq 2\epsilon.$$

By Theorem 4.1.3, there are a normal model of \mathcal{X} and arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}'$ of C^0 -type on \mathcal{X} such that

$$\overline{D} - \left(0, \sum_{P \in S} \epsilon'[P]\right) \leq \overline{\mathcal{D}} \leq \overline{D} \leq \overline{\mathcal{D}}' \leq \overline{D} + \left(0, \sum_{P \in S} \epsilon'[P]\right).$$

Then

$$0 \leq \widehat{\text{vol}}(\overline{D}) - \widehat{\text{vol}}(\overline{\mathcal{D}}) \leq \widehat{\text{vol}}(\overline{D}) - \widehat{\text{vol}}\left(\overline{D} - \left(0, \sum_{P \in S} \epsilon'[P]\right)\right)$$

and

$$0 \leq \widehat{\text{vol}}(\overline{\mathcal{D}}') - \widehat{\text{vol}}(\overline{D}) \leq \widehat{\text{vol}}\left(\overline{D} + \left(0, \sum_{P \in S} \epsilon'[P]\right)\right) - \widehat{\text{vol}}(\overline{D}).$$

On the other hand, by Proposition 5.1.2,

$$\widehat{\text{vol}}(\overline{D}) - \widehat{\text{vol}}\left(\overline{D} - \left(0, \sum_{P \in S} \epsilon'[P]\right)\right) \leq (\epsilon'/2)(d+1)[K:\mathbb{Q}]\widehat{\text{vol}}(X, D)\#(S)$$

and

$$\widehat{\text{vol}}\left(\overline{D} + \left(0, \sum_{P \in S} \epsilon'[P]\right)\right) - \widehat{\text{vol}}(\overline{D}) \leq (\epsilon'/2)(d+1)[K:\mathbb{Q}]\widehat{\text{vol}}(X, D)\#(S).$$

Thus the assertion follows. \square

The following theorem is the adelic version of arithmetic Fujita's approximation theorem. It has been already established by Boucksom and H. Chen [5]. Here we give another proof of it and generalize it to \mathbb{R} -divisors.

Theorem 5.1.6 (Fujita's approximation theorem for adelic arithmetic divisors). *Let \overline{D} be a big adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Then, for any positive number ϵ , there are a birational morphism $\mu: Y \rightarrow X$ of smooth, projective and geometrically integral varieties over K and a nef adelic arithmetic \mathbb{R} -Cartier divisor \overline{Q} of C^0 -type on Y such that $\overline{Q} \leq \mu^*(\overline{D})$ and $\widehat{\text{vol}}(\overline{Q}) \geq \widehat{\text{vol}}(\overline{D}) - \epsilon$.*

Proof. By Proposition 5.1.5, we can find a normal model \mathcal{X} of X over $\text{Spec}(O_K)$ and an arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{D}}$ of C^0 -type on \mathcal{X} such that

$$\overline{\mathcal{D}} \leq \overline{D} \quad \text{and} \quad \widehat{\text{vol}}(\overline{\mathcal{D}}) \geq \widehat{\text{vol}}(\overline{D}) - \epsilon/2.$$

Moreover, by virtue of Fujita's approximation theorem for arithmetic \mathbb{R} -Cartier divisors due to Chen-Yuan (cf. [8], [31] and [26, Theorem 5.2.2]), there are a birational morphism $\tilde{\mu}: \mathcal{Y} \rightarrow \mathcal{X}$ of generically smooth, normal and projective arithmetic varieties and a nef arithmetic \mathbb{R} -Cartier divisor $\overline{\mathcal{Q}}$ of C^0 -type on \mathcal{Y} such that

$$\overline{\mathcal{Q}} \leq \tilde{\mu}^*(\overline{\mathcal{D}}) \quad \text{and} \quad \widehat{\text{vol}}(\overline{\mathcal{Q}}) \geq \widehat{\text{vol}}(\overline{\mathcal{D}}) - \epsilon/2.$$

Thus if we set $Y = \mathcal{Y} \times_{\text{Spec}(O_K)} \text{Spec}(K)$, $\mu = \tilde{\mu}|_Y$ and $\overline{Q} = \overline{\mathcal{Q}}^a$, then we have the assertion. \square

5.2. Proof of the continuity of the volume function. The purpose of this subsection is to prove the continuity of the volume function for adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type. Namely we have the following theorem:

Theorem 5.2.1 (Continuity of the volume functions for adelic arithmetic divisors). *The volume function $\widehat{\text{vol}} : \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}} \rightarrow \mathbb{R}$ is continuous in the following sense: Let $\overline{D}_1, \dots, \overline{D}_r, \overline{A}_1, \dots, \overline{A}_{r'}$ be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Let $\{P_1, \dots, P_s\}$ be a finite subset of M_K . For a compact subset B in \mathbb{R}^r and a positive number ϵ , there are positive numbers δ and δ' such that*

$$\left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{P_l} [P_l] + \varphi_{\infty} [\infty] \right) \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \leq \epsilon$$

holds for all $a_1, \dots, a_r, \delta_1, \dots, \delta_{r'} \in \mathbb{R}$, $\varphi_{P_1} \in C^0(X_{P_1}^{\text{an}}), \dots, \varphi_{P_s} \in C^0(X_{P_s}^{\text{an}})$ and $\varphi_{\infty} \in C_{F_{\infty}}^0(X(\mathbb{C}))$ with $(a_1, \dots, a_r) \in B$, $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\sum_{l=1}^s \|\varphi_{P_l}\|_{\text{sup}} + \|\varphi_{\infty}\|_{\text{sup}} \leq \delta'$.

Proof. Let us choose a non-empty open set U of $\text{Spec}(O_K)$ such that \overline{D}_i ($i = 1, \dots, r$) and \overline{A}_j ($j = 1, \dots, r'$) have defining models $\mathcal{D}_{i,U}$ and $\mathcal{A}_{j,U}$ over U , respectively. We set $T = M_K \setminus U$ and

$$C = \max \left\{ \text{vol} \left(X, \sum_{i=1}^r a_i D_i + \sum_{j=1}^{r'} \delta_j A_j \right) \left(\sum_{i=1}^r |a_i| + 1 \right) \left| \begin{array}{l} (a_1, \dots, a_r) \in B \\ \sum_{j=1}^{r'} |\delta_j| \leq 1 \end{array} \right. \right\}.$$

We choose a positive number ϵ' such that

$$\epsilon'(d+1)[K:\mathbb{Q}]\#(T)C \leq \epsilon/3.$$

By Theorem 4.1.3, there are a normal model of \mathcal{X} and arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_r, \overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_{r'}$ of C^0 -type on \mathcal{X} such that

$$\overline{\mathcal{D}}_i^a \leq \overline{D}_i \leq \overline{\mathcal{D}}_i^a + \left(0, \sum_{P \in T} \epsilon' [P] \right) \quad \text{and} \quad \overline{\mathcal{A}}_j^a \leq \overline{A}_j \leq \overline{\mathcal{A}}_j^a + \left(0, \sum_{P \in T} \epsilon' [P] \right)$$

for all $i = 1, \dots, r$ and $j = 1, \dots, r'$. Then,

$$\begin{aligned} & \sum_{i=1}^r a_i \overline{\mathcal{D}}_i^a + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j^a - \left(0, \sum_{P \in T} \epsilon' \left(\sum_{i=1}^r |a_i| + \sum_{j=1}^{r'} |\delta_j| \right) [P] \right) \\ & \leq \sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \\ & \leq \sum_{i=1}^r a_i \overline{\mathcal{D}}_i^a + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j^a + \left(0, \sum_{P \in T} \epsilon' \left(\sum_{i=1}^r |a_i| + \sum_{j=1}^{r'} |\delta_j| \right) [P] \right). \end{aligned}$$

Therefore, by using Proposition 5.1.2,

$$(5.2.1.1) \quad \left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) \right| \\ \leq \frac{\epsilon'(d+1)[K:\mathbb{Q}]\#(T)}{2} \text{vol} \left(X, \sum_{i=1}^r a_i D_i + \sum_{j=1}^{r'} \delta_j A_j \right) \left(\sum_{i=1}^r |a_i| + \sum_{j=1}^{r'} |\delta_j| \right)$$

for all $a_1, \dots, a_r, \delta_1, \dots, \delta_{r'} \in \mathbb{R}$. Moreover, by [26, Theorem 5.2.2], there is a positive number δ such that $\delta \leq 1$ and

$$\left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) \right| \leq \epsilon/6$$

for all $(a_1, \dots, a_r) \in B$ and $\delta_1, \dots, \delta_{r'} \in \mathbb{R}$ with

$$|\delta_1| + \dots + |\delta_{r'}| \leq \delta.$$

Therefore, if we set

$$\Delta = \left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right|,$$

then

$$\Delta \leq \left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) \right| \\ + \left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \\ + \left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) \right|.$$

Thus, by using (5.2.1.1), for $(a_1, \dots, a_r) \in B$ and $\delta_1, \dots, \delta_{r'} \in \mathbb{R}$ with

$$|\delta_1| + \dots + |\delta_{r'}| \leq \delta,$$

we have

$$(5.2.1.2) \quad \Delta \leq \epsilon/6 + \epsilon/6 + \epsilon/6 = \epsilon/2.$$

On the other hand, by Proposition 5.1.2,

$$\left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{P_l} [P_l] + \varphi_\infty [\infty] \right) \right) \right. \\ \left. - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) \right|$$

$$\leq \frac{(d+1)[K:\mathbb{Q}]\text{vol}\left(X, \sum_{i=1}^r a_i D_i + \sum_{j=1}^{r'} \delta_j A_j\right)}{2} \left(\sum_{l=1}^s \|\varphi_{P_l}\|_{\text{sup}} + \|\varphi_{\infty}\|_{\text{sup}} \right).$$

Here we set

$$C' = \max \left\{ \text{vol} \left(X, \sum_{i=1}^r a_i D_i + \sum_{j=1}^{r'} \delta_j A_j \right) \mid (a_1, \dots, a_r) \in B, \sum_{j=1}^{r'} |\delta_j| \leq \delta \right\}$$

and choose a positive number δ' such that

$$(d+1)[K:\mathbb{Q}]C'\delta' \leq \epsilon.$$

Then

$$\left| \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{P_l}[P_l] + \varphi_{\infty}[\infty] \right) \right) - \widehat{\text{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) \right| \leq \epsilon/2$$

for all $a_1, \dots, a_r, \delta_1, \dots, \delta_{r'} \in \mathbb{R}$, $\varphi_{P_1} \in C^0(X_{P_1}^{\text{an}}), \dots, \varphi_{P_s} \in C^0(X_{P_s}^{\text{an}})$ and $\varphi_{\infty} \in C_{F_{\infty}}^0(X(\mathbb{C}))$ with $(a_1, \dots, a_r) \in B$, $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\sum_{l=1}^s \|\varphi_{P_l}\|_{\text{sup}} + \|\varphi_{\infty}\|_{\text{sup}} \leq \delta'$. Thus, by the above estimate together with (5.2.1.2), we have the assertion. \square

Theorem 5.2.2. *The $\hat{\chi}$ -volume function $\widehat{\text{vol}}_{\chi} : \widehat{\text{Div}}_{C^0}^{\text{a}}(X)_{\mathbb{R}} \rightarrow \mathbb{R}$ is continuous in the following sense: Let $\overline{D}_1, \dots, \overline{D}_r, \overline{A}_1, \dots, \overline{A}_{r'}$ be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Let $\{P_1, \dots, P_s\}$ be a finite subset of M_K . For $a_1, \dots, a_r \in \mathbb{R}$ and $\epsilon \in \mathbb{R}_{>0}$, there are positive numbers δ and δ' such that*

$$\left| \widehat{\text{vol}}_{\chi} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + \left(0, \sum_{l=1}^s \varphi_{P_l}[P_l] + \varphi_{\infty}[\infty] \right) \right) - \widehat{\text{vol}}_{\chi} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \leq \epsilon$$

holds for all $\delta_1, \dots, \delta_{r'} \in \mathbb{R}$, $\varphi_{P_1} \in C^0(X_{P_1}^{\text{an}}), \dots, \varphi_{P_s} \in C^0(X_{P_s}^{\text{an}})$ and $\varphi_{\infty} \in C_{F_{\infty}}^0(X(\mathbb{C}))$ with $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\sum_{l=1}^s \|\varphi_{P_l}\|_{\text{sup}} + \|\varphi_{\infty}\|_{\text{sup}} \leq \delta'$.

Proof. The proof is almost same as the proof of Theorem 5.2.1 in the case where $B = \{(a_1, \dots, a_r)\}$. We use the same notation as in the proof of Theorem 5.2.1. Then, by [16, Corollary 3.4.4], there is a positive number δ such that $\delta \leq 1$ and

$$\left| \widehat{\text{vol}}_{\chi} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i + \sum_{j=1}^{r'} \delta_j \overline{\mathcal{A}}_j \right) - \widehat{\text{vol}}_{\chi} \left(\sum_{i=1}^r a_i \overline{\mathcal{D}}_i \right) \right| \leq \epsilon/6$$

for $\delta_1, \dots, \delta_{r'} \in \mathbb{R}$ with $|\delta_1| + \dots + |\delta_{r'}| \leq \delta$. Thus, by virtue of Proposition 5.1.2, we can show the similar estimate

$$\left| \widehat{\text{vol}}_{\chi} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j \right) - \widehat{\text{vol}}_{\chi} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \leq \epsilon/2$$

as (5.2.1.2). The remaining part is exactly same as the proof of Theorem 5.2.1 by using Proposition 5.1.2. \square

5.3. Applications. Here we would like to give two applications of the continuity of the volume function, that is, the log-concavity of $\widehat{\text{vol}}$ and the generalized Hodge index theorem for adelic arithmetic divisors.

Theorem 5.3.1. *Let \overline{D}_1 and \overline{D}_2 be pseudo-effective adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Then*

$$\widehat{\text{vol}}(\overline{D}_1 + \overline{D}_2)^{1/(d+1)} \geq \widehat{\text{vol}}(\overline{D}_1)^{1/(d+1)} + \widehat{\text{vol}}(\overline{D}_2)^{1/(d+1)}.$$

Proof. Let U be a non-empty open set of $\text{Spec}(O_K)$ such that \overline{D}_1 and \overline{D}_2 have defining models $\mathcal{D}_{1,U}$ and $\mathcal{D}_{2,U}$ over U . We set $T = M_K \setminus U$. For a positive number ϵ , by Theorem 4.1.3, there are a normal model \mathcal{X} of X and arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ of C^0 -type on \mathcal{X} such that

$$\overline{D}_1 \leq \overline{\mathcal{D}}_1 \leq \overline{D}_1 + \left(0, \sum_{P \in T} \epsilon[P], 0\right) \quad \text{and} \quad \overline{D}_2 \leq \overline{\mathcal{D}}_2 \leq \overline{D}_2 + \left(0, \sum_{P \in T} \epsilon[P], 0\right).$$

As $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ are pseudo-effective, by [31, Theorem B] or [26, Theorem 5.2.2], we have

$$\widehat{\text{vol}}(\overline{\mathcal{D}}_1 + \overline{\mathcal{D}}_2)^{1/(d+1)} \geq \widehat{\text{vol}}(\overline{\mathcal{D}}_1)^{1/(d+1)} + \widehat{\text{vol}}(\overline{\mathcal{D}}_2)^{1/(d+1)},$$

and hence

$$\widehat{\text{vol}}\left(\overline{D}_1 + \overline{D}_2 + 2\epsilon \left(0, \sum_{P \in T} [P], 0\right)\right)^{1/(d+1)} \geq \widehat{\text{vol}}(\overline{D}_1)^{1/(d+1)} + \widehat{\text{vol}}(\overline{D}_2)^{1/(d+1)}.$$

Thus the assertion follows from the continuity of $\widehat{\text{vol}}$ on $\widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$ (cf. Theorem 5.2.1). \square

Theorem 5.3.2 (Generalized Hodge index theorem for adelic arithmetic divisors). *Let $\overline{D} = (D, g)$ be a relatively nef adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Then $\widehat{\deg}(\overline{D}^{d+1}) = \widehat{\text{vol}}_{\chi}(\overline{D})$. In particular, $\widehat{\deg}(\overline{D}^{d+1}) \leq \widehat{\text{vol}}(\overline{D})$. Moreover, if \overline{D} is nef, then $\widehat{\deg}(\overline{D}^{d+1}) = \widehat{\text{vol}}(\overline{D})$.*

Proof. Let \mathcal{X} be a normal model of X over $\text{Spec}(O_K)$ and $\overline{\mathcal{D}} = (\mathcal{D}, g_{\infty})$ an arithmetic \mathbb{R} -Cartier divisor of C^0 -type on \mathcal{X} . First let us see the following claim:

Claim 5.3.2.1. *If $\overline{\mathcal{D}}$ is relatively nef, then $\widehat{\deg}(\overline{\mathcal{D}}^{d+1}) = \widehat{\text{vol}}_{\chi}(\overline{\mathcal{D}})$.*

Proof. We divide the proof into four steps:

Step 1 (the case where $\overline{\mathcal{D}}$ is an arithmetic \mathbb{Q} -Cartier divisor of C^{∞} -type, \mathcal{D} is ample on \mathcal{X} and $c_1(\overline{\mathcal{D}})$ is a positive form) : This is a classic case. For example, it follows from the arithmetic Riemann-Roch theorem due to Gillet-Soulé (cf. [10]).

Step 2 (the case where $\overline{\mathcal{D}}$ is of C^{∞} -type, \mathcal{D} is relatively nef, $c_1(\overline{\mathcal{D}})$ is a semi-positive form) : As any arithmetic Cartier divisor of C^{∞} -type can be written by a difference of ample arithmetic Cartier divisors of C^{∞} -type, we can find ample arithmetic Cartier divisors $\overline{\mathcal{A}}_1, \dots, \overline{\mathcal{A}}_l$ of C^{∞} -type and real numbers a_1, \dots, a_l such that

$$\overline{\mathcal{D}} = a_1 \overline{\mathcal{A}}_1 + \dots + a_l \overline{\mathcal{A}}_l.$$

Then, for any rational numbers b_1, \dots, b_l with $a_i < b_i$ for all i , $b_1 \mathcal{A}_1 + \dots + b_l \mathcal{A}_l$ is ample and $c_1(b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l)$ is positive because

$$b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l = \overline{\mathcal{D}} + (b_1 - a_1) \overline{\mathcal{A}}_1 + \dots + (b_l - a_l) \overline{\mathcal{A}}_l.$$

Thus, by Step 1,

$$\widehat{\deg}((b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l)^{d+1}) = \widehat{\text{vol}}_\chi(b_1 \overline{\mathcal{A}}_1 + \dots + b_l \overline{\mathcal{A}}_l).$$

Therefore, the assertion follows from Theorem 5.2.2.

Step 3 (the case where $\mathcal{D} \cap X$ is ample on X): Let h be an F_∞ -invariant \mathcal{D} -Green function of C^∞ -type on $X(\mathbb{C})$ such that $c_1(\mathcal{D}, h)$ is a positive form. Then there is a continuous function ϕ on $X(\mathbb{C})$ such that $g_\infty = h + \phi$, and hence $c_1(\mathcal{D}, h) + dd^c([\phi]) \geq 0$. Thus, by [26, Lemma 4.2], there is a sequence $\{\phi_n\}_{n=1}^\infty$ of F_∞ -invariant C^∞ -functions on $X(\mathbb{C})$ with the following properties:

- (a) $\lim_{n \rightarrow \infty} \|\phi_n - \phi\|_{\text{sup}} = 0$.
- (b) If we set $\overline{\mathcal{E}}_n = (\mathcal{D}, h + \phi_n)$, then $c_1(\overline{\mathcal{E}}_n)$ is a semipositive form.

Then, by Step 2, $\widehat{\deg}(\overline{\mathcal{E}}_n^{d+1}) = \widehat{\text{vol}}_\chi(\overline{\mathcal{E}}_n)$ for all n . As $\overline{\mathcal{E}}_n = \overline{\mathcal{D}} + (0, \phi_n - \phi)$, by Theorem 5.2.2,

$$\lim_{n \rightarrow \infty} \widehat{\text{vol}}_\chi(\overline{\mathcal{E}}_n) = \widehat{\text{vol}}_\chi(\overline{\mathcal{D}}).$$

Moreover, by using [27, Lemma 1.2.1],

$$\lim_{n \rightarrow \infty} \widehat{\deg}(\overline{\mathcal{E}}_n^{d+1}) = \widehat{\deg}(\overline{\mathcal{D}}^{d+1}),$$

as required.

Step 4 (general case): Finally we prove the assertion of the claim. Let $\overline{\mathcal{A}}$ be an ample arithmetic Cartier divisor of C^∞ -type on \mathcal{X} . Then, since $\mathcal{D} + \epsilon \mathcal{A}$ is ample on X for any positive number ϵ , we have $\widehat{\deg}((\overline{\mathcal{D}} + \epsilon \overline{\mathcal{A}})^{d+1}) = \widehat{\text{vol}}_\chi(\overline{\mathcal{D}} + \epsilon \overline{\mathcal{A}})$ by Step 3. Thus, the assertion follows from Theorem 5.2.2. \square

We assume that \overline{D} is relatively nef. Let us choose a non-empty open set U of $\text{Spec}(O_K)$, a normal model \mathcal{X}_U over U and a relatively nef \mathbb{R} -Cartier divisor \mathcal{D}_U on \mathcal{X}_U such that $\mathcal{D}_U \cap X = D$ and g_P is the Green function arising from $(\mathcal{X}_U, \mathcal{D}_U)$ for all $P \in U \cap M_K$. Moreover, by Proposition 4.4.2, there is a sequence $\{(\mathcal{X}_n, \mathcal{D}_n)\}_{n=1}^\infty$ with the following properties:

- (1) \mathcal{X}_n is a normal model of X over $\text{Spec}(O_K)$ such that $\mathcal{X}_n|_U = \mathcal{X}_U$.
- (2) \mathcal{D}_n is relatively nef \mathbb{R} -Cartier divisor on \mathcal{X}_n and $\mathcal{D}_n|_U = \mathcal{D}_U$.
- (3) $\overline{D} \leq (\mathcal{D}_n, g_\infty)^a$.
- (4) If we set $\phi_{n,P} = g_{((\mathcal{X}_n)_{(P)}, (\mathcal{D}_n)_{(P)})} - g_P$ for $P \in M_K \setminus U$, then

$$\lim_{n \rightarrow \infty} \|\phi_{n,P}\|_{\text{sup}} = 0.$$

As $(\mathcal{D}_n, g_\infty)^a = \overline{D} + (0, \sum_{P \in M_K \setminus U} \phi_{n,P} [P])$, by Theorem 5.2.1 and Theorem 5.2.2,

$$(5.3.2.2) \quad \lim_{n \rightarrow \infty} \widehat{\text{vol}}((\mathcal{D}_n, g_\infty)) = \widehat{\text{vol}}(\overline{D}) \quad \text{and} \quad \lim_{n \rightarrow \infty} \widehat{\text{vol}}_\chi((\mathcal{D}_n, g_\infty)) = \widehat{\text{vol}}_\chi(\overline{D}).$$

Here let us see

$$(5.3.2.3) \quad \lim_{n \rightarrow \infty} \widehat{\deg}((\mathcal{D}_n, g_\infty)^{d+1}) = \widehat{\deg}(\overline{D}^{d+1}).$$

Indeed, we set

$$\psi_P = g_P - g_{((\mathcal{X}_1)_{(P)}, (\mathcal{D}_1)_{(P)})} \quad \text{and} \quad \psi_{n,P} = g_{((\mathcal{X}_n)_{(P)}, (\mathcal{D}_n)_{(P)})} - g_{((\mathcal{X}_1)_{(P)}, (\mathcal{D}_1)_{(P)})}$$

for $P \in M_K \setminus U$. Note that $\phi_{n,P} = \psi_P - \psi_{n,P}$. Then, by using Lemma 1.3.4, we can see

$$\begin{aligned} \widehat{\deg}((\mathcal{D}_n, g_\infty)^{d+1}) &= \widehat{\deg}((\mathcal{D}_1, g_\infty)^{d+1}) \\ &\quad + \sum_{i=1}^{d+1} \sum_{P \in M_K \setminus U} \log \#(O_K/P) \widehat{\deg}_P((\mathcal{D}_n)_{(P)}^{i-1} \cdot (\mathcal{D}_1)_{(P)}^{d+1-i}; \psi_{n,P}). \end{aligned}$$

Thus, by virtue of Proposition 2.4.3,

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{\deg}((\mathcal{D}_n, g_\infty)^{d+1}) &= \widehat{\deg}((\mathcal{D}_1, g_\infty)^{d+1}) \\ &\quad + \sum_{i=1}^{d+1} \sum_{P \in M_K \setminus U} \log \#(O_K/P) \widehat{\deg}_P((D, g_P)^{i-1} \cdot (\mathcal{D}_1)_{(P)}^{d+1-i}; \psi_P) = \widehat{\deg}(\overline{D}^{d+1}), \end{aligned}$$

as desired.

By (5.3.2.2) and (5.3.2.3) together with the above claim, we have the first assertion. If \overline{D} is nef, then $(\mathcal{D}_n, g_\infty)$ is also nef by the property (3) and Lemma 4.2.3, and hence the second assertion follows from (5.3.2.2) and (5.3.2.3) by using [26, Proposition 6.4.2] and [27, Proposition 2.1.1]. \square

6. ZARISKI DECOMPOSITIONS OF ADELIC ARITHMETIC DIVISORS ON ARITHMETIC SURFACES

Let \mathcal{X} be a regular projective arithmetic surface and let $\overline{\mathcal{D}}$ be an arithmetic \mathbb{R} -Cartier divisor of C^0 -type on \mathcal{X} . The set of all nef arithmetic \mathbb{R} -Cartier divisors $\overline{\mathcal{D}}$ of C^0 -type on \mathcal{X} with $\overline{\mathcal{D}} \leq \overline{\mathcal{D}}$ is denoted by $\Upsilon(\overline{\mathcal{D}})$. In [26, Theorem 9.2.1], it is shown that if $\Upsilon(\overline{D}) \neq \emptyset$, then $\Upsilon(\overline{D})$ has the greatest element $\overline{\mathcal{D}}$, that is, $\overline{\mathcal{D}} \in \Upsilon(\overline{D})$ and $\overline{\mathcal{D}} \leq \overline{\mathcal{D}}$ for all $\overline{\mathcal{D}} \in \Upsilon(\overline{D})$. If we set $\overline{\mathcal{N}} := \overline{\mathcal{D}} - \overline{\mathcal{D}}$, then $\overline{\mathcal{D}} = \overline{\mathcal{D}} + \overline{\mathcal{N}}$ yields the Zariski decomposition of $\overline{\mathcal{D}}$. For example, we can see that the natural map

$$\hat{H}^0(\mathcal{X}, n\overline{\mathcal{D}}) \rightarrow \hat{H}^0(\mathcal{X}, n\overline{\mathcal{D}})$$

is bijective for every $n \in \mathbb{Z}_{>0}$. In particular, $\widehat{\text{vol}}(\overline{\mathcal{D}}) = \widehat{\text{vol}}(\overline{\mathcal{D}})$, so that it gives rise to a refinement of Fujita's approximation theorem for arithmetic divisors. In this section, we consider a generalization of the above result to an adelic arithmetic \mathbb{R} -Cartier divisor.

6.1. Local Zariski decompositions of adelic divisors on algebraic curves. Let k be a field and v a non-trivial discrete valuation of k . We assume that k° is excellent. Let ϖ be a uniformizing parameter of k° . Let X be a projective, smooth and geometrically integral curve over k . Let k_v be the completion of k with respect to v and $X_v := X \times_{\text{Spec}(k)} \text{Spec}(k_v)$. The purpose of this subsection is to prove the following theorem:

Theorem 6.1.1 (Local Zariski decomposition). *Let $\overline{D} = (D, g)$ be an adelic \mathbb{R} -Cartier divisor on X and let Q be an \mathbb{R} -Cartier divisor on X with $Q \leq D$. Here we set*

$$\Sigma(\overline{D}; Q) := \left\{ \overline{L} \mid \begin{array}{l} \overline{L} \text{ is a relatively nef adelic } \mathbb{R}\text{-Cartier divisor on } X \\ \text{such that } L \leq Q \text{ and } \overline{L} \leq \overline{D} \end{array} \right\}.$$

We assume that $\deg(Q) \geq 0$. Then there exists a Q -Green function q of $(C^0 \cap \text{PSH})$ -type on X_v^{an} such that $\overline{Q} := (Q, q)$ gives rise to the greatest element of $\Sigma(\overline{D}; Q)$, that is, $\overline{Q} \in \Sigma(\overline{D}; Q)$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Sigma(\overline{D}; Q)$. Moreover, we have the following:

- (1) *If \overline{D} is given by an \mathbb{R} -Cartier divisor \mathcal{D} on a regular model \mathcal{X} of X over $\text{Spec}(k^\circ)$, then \overline{Q} is given by a relatively nef \mathbb{R} -Cartier divisor \mathcal{Q} on \mathcal{X} .*

(2) If $Q = D$, then $\widehat{\deg}_v(\bar{Q}; g - q) = 0$ (cf. Proposition-Definition 2.4.3).

Before starting the proof of Theorem 6.1.1, we need to prepare two lemmas.

Lemma 6.1.2. *Let \mathcal{X} be a regular model of X over $\text{Spec}(k^\circ)$. Then we have the following:*

- (1) *Let $\pi : \mathcal{X}' \rightarrow \mathcal{X}$ be a birational morphism of regular models of X over $\text{Spec}(k^\circ)$, and let \mathcal{L}' be a relatively nef \mathbb{R} -Cartier divisor on \mathcal{X}' . Then $\pi_*(\mathcal{L}')$ is relatively nef and $\pi^*(\pi_*(\mathcal{L}')) - \mathcal{L}'$ is effective.*
- (2) *If $\mathcal{L}_1, \dots, \mathcal{L}_l$ are relatively nef \mathbb{R} -Cartier divisors on \mathcal{X} , then*

$$\max\{\mathcal{L}_1, \dots, \mathcal{L}_l\}$$

is also relatively nef (for the definition of $\max\{\mathcal{L}_1, \dots, \mathcal{L}_l\}$, see Conventions and terminology 0.5.8).

- (3) *Let \mathcal{D} be an \mathbb{R} -Cartier divisor on \mathcal{X} and $g = g_{(\mathcal{X}, \mathcal{D})}$. Then g is of $(C^0 \cap \text{PSH})$ -type if and only if \mathcal{D} is relatively nef.*

Proof. (1) Let C be an irreducible component of the central fiber \mathcal{X}_\circ of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$. Then

$$0 \leq (\mathcal{L}' \cdot \pi^*(C)) = (\pi_*(\mathcal{L}') \cdot C).$$

Thus $\pi_*(\mathcal{L}')$ is relatively nef.

Let us consider the second assertion. By [19, Theorem 9.2.2], π can be obtained by a succession of blowing-ups at closed points. We prove it by induction on the number of blowing-ups. First we consider the case where π is a blowing-up at a closed point. Let C be the exceptional curve of π . Then

$$\pi^*(\pi_*(\mathcal{L}')) - \mathcal{L}' = aC$$

for some $a \in \mathbb{R}$. As

$$((\pi^*(\pi_*(\mathcal{L}')) - \mathcal{L}') \cdot C) = -(\mathcal{L}' \cdot C) \leq 0 \quad \text{and} \quad (C \cdot C) < 0,$$

we have $a \geq 0$, as required. In general, we decompose π into two birational morphisms $\pi_1 : \mathcal{X}' \rightarrow \mathcal{X}_1$ and $\pi_2 : \mathcal{X}_1 \rightarrow \mathcal{X}$ of regular models of X , that is, $\pi = \pi_2 \circ \pi_1$. Note that $(\pi_1)_*(\mathcal{L}')$ is relatively nef by the previous observation. Thus, by the induction hypothesis,

$$\pi_1^*(\pi_1)_*(\mathcal{L}') - \mathcal{L}' \quad \text{and} \quad \pi_2^*(\pi_2)_*((\pi_1)_*(\mathcal{L}')) - (\pi_1)_*(\mathcal{L}')$$

are effective, so that

$$\pi^*(\pi_*(\mathcal{L}')) - \pi_1^*(\pi_1)_*(\mathcal{L}') = \pi_1^*(\pi_2^*(\pi_2)_*((\pi_1)_*(\mathcal{L}')) - (\pi_1)_*(\mathcal{L}'))$$

is also effective. Therefore, as

$$\pi^*(\pi_*(\mathcal{L}')) - \mathcal{L}' = (\pi^*(\pi_*(\mathcal{L}')) - \pi_1^*(\pi_1)_*(\mathcal{L}')) + (\pi_1^*(\pi_1)_*(\mathcal{L}') - \mathcal{L}'),$$

we have the assertion.

(2) We set $\mathcal{L}'_i := \max\{\mathcal{L}_1, \dots, \mathcal{L}_l\} - \mathcal{L}_i$ for each i . Let C be an irreducible component of \mathcal{X}_\circ . Then there is i such that $C \not\subseteq \text{Supp}_{\mathbb{R}}(\mathcal{L}'_i)$. As \mathcal{L}'_i is effective, we have $\deg(\mathcal{L}'_i|_C) \geq 0$, so that

$$\deg(\max\{\mathcal{L}_1, \dots, \mathcal{L}_l\}|_C) = \deg(\mathcal{L}_i|_C) + \deg(\mathcal{L}'_i|_C) \geq 0.$$

(3) This is a special case of Proposition 2.1.7. □

Lemma 6.1.3. *Let \mathcal{X} be a regular model of X and \mathcal{D} an \mathbb{R} -Cartier divisor on \mathcal{X} . Let Q be an \mathbb{R} -Cartier divisor on X with $Q \leq D := \mathcal{D} \cap X$. Here we set*

$$\Sigma_{\mathcal{X}}(\mathcal{D}; Q) := \left\{ \mathcal{L} \mid \begin{array}{l} \mathcal{L} \text{ is a relatively nef } \mathbb{R}\text{-Cartier divisor on } \mathcal{X} \\ \text{such that } \mathcal{L} \cap X \leq Q \text{ and } \mathcal{L} \leq \mathcal{D} \end{array} \right\}.$$

- (1) *We assume that $\deg(Q) \geq 0$. Then there is a relatively nef \mathbb{R} -Cartier divisor \mathcal{Q} on \mathcal{X} such that $\mathcal{Q} \cap X = Q$ and \mathcal{Q} gives rise to the greatest element of $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$, that is, $\mathcal{Q} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ and $\mathcal{L} \leq \mathcal{Q}$ for all $\mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$. Moreover, if $Q = D$, then $(\mathcal{Q} \cdot \mathcal{D} - \mathcal{Q}) = 0$, that is, $\widehat{\deg}_v(\mathcal{Q}^a; g_{(\mathcal{X}, \mathcal{D} - \mathcal{Q})}) = 0$.*
- (2) *Let $\pi : \mathcal{X}' \rightarrow \mathcal{X}$ be a birational morphism of regular models of X . If \mathcal{Q} is the greatest element of $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$, then $\pi^*(\mathcal{Q})$ is also the greatest element of*

$$\Sigma_{\mathcal{X}'}(\pi^*(\mathcal{D}); Q).$$

Proof. (1) Let us begin with following claim:

Claim 6.1.3.1. (i) *There is a relatively nef \mathbb{R} -Cartier divisor \mathcal{P}_0 on \mathcal{X} with $\mathcal{P}_0 \cap X = Q$.*

(ii) *There is $\mathcal{P} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ with $\mathcal{P} \cap X = Q$.*

Proof. (i) First we assume that $\deg(Q) = 0$. Let \mathcal{P}' be the closure of Q in \mathcal{X} . Let C_1, \dots, C_r be irreducible components of \mathcal{X}_\circ . As $(\mathcal{P}' \cdot \mathcal{X}_\circ) = 0$, by Zariski's lemma, we can find $a_1, \dots, a_r \in \mathbb{R}$ such that

$$\left(\sum_{i=1}^r a_i C_i \cdot C_j \right) = (\mathcal{P}' \cdot C_j)$$

for all $j = 1, \dots, r$. Thus, if we set $\mathcal{P}_0 = \mathcal{P}' - \sum_{i=1}^r a_i C_i$, then \mathcal{P}_0 is relatively nef and $\mathcal{P}_0 \cap X = Q$.

Next we assume that $\deg(Q) > 0$. Then there is $\phi \in \text{Rat}(\mathcal{X})_{\mathbb{Q}}^\times$ such that $Q + (\phi)_X \geq 0$, where $(\phi)_X$ is the \mathbb{Q} -principal divisor of ϕ on X . Let \mathcal{P}' be the closure of $Q + (\phi)_X$ in \mathcal{X} . As $Q + (\phi)_X$ is effective, \mathcal{P}' is relatively nef. Here we set $\mathcal{P}_0 = \mathcal{P}' - (\phi)$ on \mathcal{X} . Then \mathcal{P}_0 is relatively nef and

$$\mathcal{P}_0 \cap X = \mathcal{P}' \cap X - (\phi) \cap X = Q + (\phi)_X - (\phi)_X = Q.$$

(ii) follows from (i) because $\mathcal{P}_0 - n\mathcal{X}_\circ \leq \mathcal{D}$ for a sufficiently large n and $\mathcal{P}_0 - n\mathcal{X}_\circ$ is relatively nef. \square

For a prime divisor C on \mathcal{X} (that is, C is a reduced and irreducible curve on \mathcal{X}), we set

$$q_C := \sup \{ \text{mult}_C(\mathcal{L}) \mid \mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q) \},$$

which exists in \mathbb{R} because $\text{mult}_C(\mathcal{L}) \leq \text{mult}_C(\mathcal{D})$ for all $\mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$. We fix $\mathcal{P} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ with $\mathcal{P} \cap X = Q$ by using Claim 6.1.3.1.

Claim 6.1.3.2. *There is a sequence $\{\mathcal{L}_n\}_{n=1}^\infty$ of \mathbb{R} -Cartier divisors in $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ such that $\mathcal{P} \leq \mathcal{L}_n$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} \text{mult}_C(\mathcal{L}_n) = q_C$ for all prime divisors C in $\text{Supp}_{\mathbb{R}}(\mathcal{D}) \cup \text{Supp}_{\mathbb{R}}(\mathcal{P})$.*

Proof. For each prime divisor C in $\text{Supp}_{\mathbb{R}}(\mathcal{D}) \cup \text{Supp}_{\mathbb{R}}(\mathcal{P})$, there is a sequence $\{\mathcal{L}_{C,n}\}_{n=1}^\infty$ in $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ such that

$$\lim_{n \rightarrow \infty} \text{mult}_C(\mathcal{L}_{C,n}) = q_C.$$

If we set

$$\mathcal{L}_n = \max \left(\{ \mathcal{L}_{C,n} \}_{C \subseteq \text{Supp}_{\mathbb{R}}(\mathcal{D}) \cup \text{Supp}_{\mathbb{R}}(\mathcal{P})} \cup \{ \mathcal{P} \} \right),$$

then $\mathcal{P} \leq \mathcal{L}_n$ and $\mathcal{L}_n \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ by (2) in Lemma 6.1.2. Moreover, as

$$\text{mult}_C(\mathcal{L}_{C,n}) \leq \text{mult}_C(\mathcal{L}_n) \leq q_C,$$

$$\lim_{n \rightarrow \infty} \text{mult}_C(\mathcal{L}_n) = q_C. \quad \square$$

Since $\max\{\mathcal{P}, \mathcal{L}\} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ for all $\mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ by (2) in Lemma 6.1.2, we have

$$\text{mult}_C(\mathcal{P}) \leq q_C \leq \text{mult}_C(\mathcal{D}).$$

In particular, if $C \not\subseteq \text{Supp}_{\mathbb{R}}(\mathcal{D}) \cup \text{Supp}_{\mathbb{R}}(\mathcal{P})$, then $q_C = 0$, so that we can set $\mathcal{Q} := \sum_C q_C C$. Clearly $\mathcal{Q} \cap X = Q$ because $\mathcal{P} \leq \mathcal{Q}$. Moreover, $\mathcal{Q} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ by Claim 6.1.3.2, and $\mathcal{L} \leq \mathcal{Q}$ for all $\mathcal{L} \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$ by our construction.

Here we assume that $Q = D$. Then $\mathcal{D} - \mathcal{Q}$ is vertical. Let C be an irreducible component of $\text{Supp}_{\mathbb{R}}(\mathcal{D} - \mathcal{Q})$. If $(\mathcal{Q} \cdot C) > 0$, then $\mathcal{Q} + \epsilon C$ is relatively nef for a sufficiently small positive number ϵ . Moreover, $\mathcal{Q} + \epsilon C \leq \mathcal{D}$. This is a contradiction, so that $(\mathcal{Q} \cdot C) = 0$. Therefore, $(\mathcal{Q} \cdot \mathcal{D} - \mathcal{Q}) = 0$.

(2) Clearly $\pi^*(\mathcal{Q}) \in \Sigma_{\mathcal{X}'}(\pi^*(\mathcal{D}); Q)$. Let $\mathcal{L}' \in \Sigma_{\mathcal{X}'}(\pi^*(\mathcal{D}); Q)$. As $\pi_*(\mathcal{L}')$ is relatively nef by (1) in Lemma 6.1.2, we have $\pi_*(\mathcal{L}') \in \Sigma_{\mathcal{X}}(\mathcal{D}; Q)$, and hence $\pi_*(\mathcal{L}') \leq \mathcal{Q}$. Thus, by using (1) in Lemma 6.1.2,

$$\mathcal{L}' \leq \pi^*(\pi_*(\mathcal{L}')) \leq \pi^*(\mathcal{Q}),$$

as required. \square

Proof of Theorem 6.1.1. Let us start the proof of Theorem 6.1.1. We fix a regular model \mathcal{X}_0 with the following properties:

- (a) If \overline{D} is given by an \mathbb{R} -Cartier divisor \mathcal{D} on a regular model \mathcal{X} , then $\mathcal{X}_0 = \mathcal{X}$.
- (b) There is a relatively nef \mathbb{R} -Cartier divisor \mathcal{Q}_0 on \mathcal{X}_0 with $\mathcal{Q}_0 \cap X = Q$ (for details, see Claim 6.1.3.1).

By Theorem 3.3.7, for each $n \geq 1$, we can find a regular model \mathcal{X}_n and an \mathbb{R} -Cartier divisor \mathcal{D}_n on \mathcal{X}_n such that

$$\overline{D} - \frac{1}{n(n+1)}(\mathcal{X}_n)_\circ^a \leq \mathcal{D}_n^a \leq \overline{D} + \frac{1}{n(n+1)}(\mathcal{X}_n)_\circ^a.$$

Replacing \mathcal{X}_n by a suitable regular model of X if necessarily, we may assume that there is a birational morphism $\pi_{n+1} : \mathcal{X}_{n+1} \rightarrow \mathcal{X}_n$ for every $n \geq 0$. Note that if \overline{D} is given by an \mathbb{R} -Cartier divisor \mathcal{D} on \mathcal{X} , then $\mathcal{X}_n = \mathcal{X}$ and $\mathcal{D}_n = \mathcal{D}$ for all $n \geq 1$. By using (1) in Lemma 6.1.3, let \mathcal{Q}_n be the greatest element of $\Sigma_{\mathcal{X}_n}(\mathcal{D}_n; Q)$. Let us check the following claim:

Claim 6.1.4. (i) *The following inequalities*

$$\mathcal{D}_{n+1} - \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ \leq \pi_{n+1}^*(\mathcal{Q}_n) \leq \mathcal{D}_{n+1} + \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ$$

hold for all $n \geq 1$.

(ii) *Moreover,*

$$\mathcal{Q}_{n+1} - \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ \leq \pi_{n+1}^*(\mathcal{Q}_n) \leq \mathcal{Q}_{n+1} + \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ$$

hold for all $n \geq 1$.

Proof. (i) The first inequality follows from the following observation:

$$\begin{aligned}\pi_{n+1}^*(\mathcal{Q}_n)^a &\geq \overline{D} - \frac{1}{n(n+1)}\pi_{n+1}^*((\mathcal{X}_n)_\circ)^a \\ &\geq \left(\mathcal{Q}_{n+1}^a - \frac{1}{(n+1)(n+2)}(\mathcal{X}_{n+1})_\circ^a \right) - \frac{1}{n(n+1)}(\mathcal{X}_{n+1})_\circ^a \\ &\geq \left(\mathcal{Q}_{n+1} - \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ \right)^a.\end{aligned}$$

The second inequality is similar.

(ii) Note that $\pi_{n+1}^*(\mathcal{Q}_n) - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \in \Sigma_{\mathcal{X}_{n+1}}(\mathcal{Q}_{n+1}; Q)$ because

$$\pi_{n+1}^*(\mathcal{Q}_n) - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \leq \mathcal{Q}_{n+1}$$

by (i), so that $\pi_{n+1}^*(\mathcal{Q}_n) - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \leq \mathcal{Q}_{n+1}$. Similarly

$$\mathcal{Q}_{n+1} - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \in \Sigma_{\mathcal{X}_{n+1}}(\pi_{n+1}^*(\mathcal{Q}_n); Q)$$

by using (i), and hence $\mathcal{Q}_{n+1} - (2/n(n+2))(\mathcal{X}_{n+1})_\circ \leq \pi_{n+1}^*(\mathcal{Q}_n)$ by (2) in Lemma 6.1.3. \square

Let $\mathcal{E}_n := \mathcal{Q}_n - \rho_n^*(\mathcal{Q}_0)$, where $\rho_n := \pi_1 \circ \cdots \circ \pi_n : \mathcal{X}_n \rightarrow \mathcal{X}_0$. Then \mathcal{E}_n is vertical and

$$\mathcal{E}_{n+1} - \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ \leq \pi_{n+1}^*(\mathcal{E}_n) \leq \mathcal{E}_{n+1} + \frac{2}{n(n+1)}(\mathcal{X}_{n+1})_\circ$$

by (ii) of the previous claim, so that

$$\varphi_{n+1} - 4 \left(\frac{1}{n} - \frac{1}{n+1} \right) (-\log v(\varpi)) \leq \varphi_n \leq \varphi_{n+1} + 4 \left(\frac{1}{n} - \frac{1}{n+1} \right) (-\log v(\varpi)),$$

where $\varphi_n := g_{(\mathcal{X}_n, \mathcal{E}_n)}$. Therefore, if we set

$$\varphi'_n = \varphi_n - \frac{4(-\log v(\varpi))}{n} \quad \text{and} \quad \varphi''_n = \varphi_n + \frac{4(-\log v(\varpi))}{n},$$

then

$$\varphi'_1 \leq \cdots \leq \varphi'_n \leq \varphi'_{n+1} \leq \cdots \leq \varphi''_{n+1} \leq \varphi''_n \leq \cdots \leq \varphi''_1,$$

and hence $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$ exists for each $x \in X_v^{\text{an}}$. Moreover, as

$$|\varphi_n(x) - \varphi(x)| \leq \varphi''_n(x) - \varphi'_n(x) \leq (8/n)(-\log v(\varpi)),$$

the sequence $\{\varphi_n\}_{n=1}^\infty$ converges to φ uniformly. In particular, φ is continuous on X_v^{an} . We set $q := g_{(\mathcal{X}_0, \mathcal{Q}_0)} + \varphi$. As \mathcal{Q}_n is relatively nef, q is a Q -Green function of (PSH $\cap C^0$)-type. Note that in the case where \overline{D} is given by a relatively nef \mathbb{R} -Cartier divisor \mathcal{Q} on \mathcal{X} , then $q = g_{(\mathcal{X}, \mathcal{Q}_1)}$.

Let us see that $\overline{Q} := (Q, q)$ is the greatest element of $\Sigma(\overline{D}; Q)$. As $\{\varphi_n\}_{n=1}^\infty$ converges φ uniformly and

$$g_{(\mathcal{X}_0, \mathcal{Q}_0)} + \varphi_n = g_{(\mathcal{X}_n, \mathcal{Q}_n)} \leq g_{(\mathcal{X}_n, \mathcal{Q}_n)} \leq g + \frac{-2\log v(\varpi)}{n(n+1)},$$

we can see that $\overline{Q} \leq \overline{D}$, that is, $\overline{Q} \in \Sigma(\overline{D}; Q)$, so that we need to see that $\overline{L} \leq \overline{Q}$ for all $\overline{L} = (L, g_L) \in \Sigma(\overline{D}; Q)$.

First we assume that \overline{L} is given by an \mathbb{R} -Cartier divisor \mathcal{L} on a regular model \mathcal{Y} . By (3) in Lemma 6.1.2, \mathcal{L} is relatively nef. For each $n \geq 1$, we choose a regular model \mathcal{Y}_n

of X such that there are birational morphisms $v_n : \mathcal{Y}_n \rightarrow \mathcal{Y}$ and $\mu_n : \mathcal{Y}_n \rightarrow \mathcal{X}_n$. If we set $\mathcal{F}_n = (\mu_n)_*(v_n^*(\mathcal{L}))$, then \mathcal{F}_n is relatively nef by (1) in Lemma 6.1.2. Moreover, as

$$v_n^*(\mathcal{L})^a \leq \overline{D} \leq \left(\mu_n^*(\mathcal{Q}_n) + \frac{1}{n(n+1)}(\mathcal{Y}_n)_o \right)^a,$$

we have $\mathcal{F}_n \leq \mathcal{Q}_n + (1/n(n+1))(\mathcal{X}_n)_o$ by using Proposition 2.2.1, and hence

$$\mathcal{F}_n \leq \mathcal{Q}_n + (1/n(n+1))(\mathcal{X}_n)_o$$

because $\mathcal{F}_n - (1/n(n+1))(\mathcal{X}_n)_o \in \Sigma_{\mathcal{X}_n}(\mathcal{Q}_n; \mathbb{Q})$. Therefore, by (1) in Lemma 6.1.2,

$$v_n^*(\mathcal{L}) \leq \mu_n^*(\mathcal{F}_n) \leq \mu_n^*(\mathcal{Q}_n) + \frac{1}{n(n+1)}(\mathcal{Y}_n)_o.$$

In particular,

$$g_{(\mathcal{Y}, \mathcal{L})} \leq g_{(\mathcal{X}_n, \mathcal{Q}_n)} + \frac{-2 \log v(\varpi)}{n(n+1)} = g_{(\mathcal{X}_0, \mathcal{Q}_0)} + \varphi_n + \frac{-2 \log v(\varpi)}{n(n+1)}.$$

Note that $\{\varphi_n\}_{n=1}^\infty$ converges φ uniformly, so that we have $g_{(\mathcal{Y}, \mathcal{L})} \leq q$.

In general, by Proposition 2.1.8, there are a sequence $\{\mathcal{Y}_l\}_{l=1}^\infty$ of regular models of X and a sequence $\{\mathcal{L}_l\}_{l=1}^\infty$ of relatively nef \mathbb{R} -Cartier divisors such that \mathcal{L}_l is defined on \mathcal{Y}_l , $\mathcal{L}_l \cap X = L$, $(\mathcal{L}_l)^a \leq \overline{L}$ and $g_L = \lim_{l \rightarrow \infty} g_{(\mathcal{Y}_l, \mathcal{L}_l)}$ uniformly. By the previous observation, $g_{(\mathcal{Y}_l, \mathcal{L}_l)} \leq q$ for all l , so that $g_L \leq q$.

Let us see the additional assertions (1) and (2) in the theorem. The assertion (1) is obvious by our construction. Let us consider (2). We assume that $Q = D$. If we set $\theta_n := g_{(\mathcal{X}_n, \mathcal{Q}_n)} - g_{(\mathcal{X}_n, \mathcal{Q}_n)}$ and $\theta = g - q$, then $\{\theta_n\}_{n=1}^\infty$ converges to θ uniformly because

$$\theta_n = (g_{(\mathcal{X}_n, \mathcal{Q}_n)} - g) + g - (g_{(\mathcal{X}_0, \mathcal{Q}_0)} + \varphi_n).$$

Thus, by Proposition-Definition 2.4.3,

$$\lim_{n \rightarrow \infty} \widehat{\deg}_v(\mathcal{Q}_n; \theta_n) = \widehat{\deg}_v(\overline{Q}; \theta).$$

On the other hand, $\widehat{\deg}_v(\mathcal{Q}_n; \theta_n) = 0$ by Lemma 6.1.3, so that the assertion (2) follows. \square

Finally we consider the maximal of two Green functions, which will be used in the next subsection.

Proposition 6.1.5. *Let D_1 and D_2 be \mathbb{R} -Cartier divisors on X and let $D_3 := \max\{D_1, D_2\}$. For $i = 1, 2$, let g_i be a D_i -Green function of C^0 -type on X_v^{an} .*

- (1) *$\max\{g_1, g_2\}$ is a D_3 -Green function of C^0 -type on X_v^{an} .*
- (2) *If g_1 and g_2 are of $(C^0 \cap \text{PSH})$ -type, then $\max\{g_1, g_2\}$ is also of $(C^0 \cap \text{PSH})$ -type.*

Proof. (1) Let $\pi : X_v \rightarrow X$ be the canonical morphism. It is easy to see that

$$\max\{\pi^*(D_1), \pi^*(D_2)\} = \pi^*(\max\{D_1, D_2\}),$$

so that the assertion follows from Proposition 2.1.5.

(2) For each $n \geq 1$, by Proposition 2.1.8, there are a regular model \mathcal{X}'_n of X and relatively nef \mathbb{R} -Cartier divisors $\mathcal{Q}'_{1,n}$ and $\mathcal{Q}'_{2,n}$ on \mathcal{X}'_n such that

$$0 \leq g_1 - g_{(\mathcal{X}'_n, \mathcal{Q}'_{1,n})} \leq 1/n \quad \text{and} \quad 0 \leq g_2 - g_{(\mathcal{X}'_n, \mathcal{Q}'_{2,n})} \leq 1/n.$$

On the other hand, by Theorem 3.3.7, we can find a regular model \mathcal{X}''_n of X and an \mathbb{R} -Cartier divisors \mathcal{E}''_n on \mathcal{X}''_n such that $\mathcal{E}''_n \cap X = D_3$ and

$$\max\{g_1, g_2\} \leq g_{(\mathcal{X}''_n, \mathcal{E}''_n)} \leq \max\{g_1, g_2\} + 1/n.$$

We choose birational morphisms $v_n : \mathcal{X}_n \rightarrow \mathcal{X}'_n$ and $\mu_n : \mathcal{X}_n \rightarrow \mathcal{X}''_n$ and we set

$$\mathcal{D}_{1,n} := v_n^*(\mathcal{D}'_{1,n}), \mathcal{D}_{2,n} := v_n^*(\mathcal{D}'_{2,n}), \mathcal{D}_{3,n} := \max\{\mathcal{D}_{1,n}, \mathcal{D}_{2,n}\} \text{ and } \mathcal{E}_n := \mu_n^*(\mathcal{E}''_n).$$

As $\mathcal{D}_{1,n} \leq \mathcal{D}_{3,n}$ and $\mathcal{D}_{2,n} \leq \mathcal{D}_{3,n}$, we have

$$\max\{g_{(\mathcal{X}_n, \mathcal{D}_{1,n})}, g_{(\mathcal{X}_n, \mathcal{D}_{2,n})}\} \leq g_{(\mathcal{X}_n, \mathcal{D}_{3,n})},$$

so that

$$\begin{aligned} \max\{g_1, g_2\} - 1/n &= \max\{g_1 - 1/n, g_2 - 1/n\} \\ &\leq \max\{g_{(\mathcal{X}_n, \mathcal{D}_{1,n})}, g_{(\mathcal{X}_n, \mathcal{D}_{2,n})}\} \leq g_{(\mathcal{X}_n, \mathcal{D}_{3,n})}. \end{aligned}$$

Moreover,

$$g_{(\mathcal{X}_n, \mathcal{D}_{1,n})} \leq g_1 \leq \max\{g_1, g_2\} \leq g_{(\mathcal{X}_n, \mathcal{E}_n)}$$

and

$$g_{(\mathcal{X}_n, \mathcal{D}_{2,n})} \leq g_2 \leq \max\{g_1, g_2\} \leq g_{(\mathcal{X}_n, \mathcal{E}_n)},$$

and hence, by Proposition 2.2.1,

$$\mathcal{D}_{1,n} \leq \mathcal{E}_n \quad \text{and} \quad \mathcal{D}_{2,n} \leq \mathcal{E}_n,$$

so that $\mathcal{D}_{3,n} \leq \mathcal{E}_n$. Therefore,

$$\max\{g_1, g_2\} - 1/n \leq g_{(\mathcal{X}_n, \mathcal{D}_{3,n})} \leq \max\{g_1, g_2\} + 1/n.$$

Note that $\mathcal{D}_{3,n}$ is relatively nef by (2) in Lemma 6.1.2, and hence $\max\{g_1, g_2\}$ is of $(C^0 \cap \text{PSH})$ -type. \square

6.2. Proof of Zariski decompositions for adelic arithmetic divisors. In this subsection, we give the proof of Zariski decompositions for adelic arithmetic divisors. Let X be a projective, smooth and geometrically integral curve over a number field K . Let us begin with decompositions for global adelic divisors.

Theorem 6.2.1. *Let $\overline{D} = (D, \{g_P\}_{P \in M_K})$ be a global adelic \mathbb{R} -Cartier divisor on X (cf. Definition 4.1.1) and let Q be an \mathbb{R} -Cartier divisor on X with $Q \leq D$. Here we set*

$$\Sigma(\overline{D}; Q) := \left\{ \overline{L} = (L, \{l_P\}_{P \in M_K}) \left| \begin{array}{l} \overline{L} \text{ is a relatively nef global adelic } \mathbb{R}\text{-Cartier} \\ \text{divisor on } X \text{ such that } L \leq Q \text{ and } \overline{L} \leq \overline{D} \end{array} \right. \right\}.$$

If $\deg(Q) \geq 0$, then there exists a Q -Green function q_P of $(\text{PSH} \cap C^0)$ -type on X_P^{an} for each $P \in M_K$ such that $\overline{Q} := (Q, \{q_P\}_{P \in M_K})$ gives rise to the greatest element of $\Sigma(\overline{D}; Q)$, that is, $\overline{Q} \in \Sigma(\overline{D}; Q)$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Sigma(\overline{D}; Q)$. Moreover, if there are a non-empty Zariski open set U of $\text{Spec}(O_K)$, a regular model \mathcal{X}_U of X over U and an \mathbb{R} -Cartier divisor \mathcal{D}_U on \mathcal{X}_U such that g_P is the Green function arising from \mathcal{D}_U for all $P \in U \cap M_K$, then there is a relatively nef \mathbb{R} -Cartier divisor \mathcal{Q}_U on \mathcal{X}_U such that q_P is given by \mathcal{Q}_U for all $P \in U \cap M_K$.

Proof. Let us choose a non-empty Zariski open set U of $\text{Spec}(O_K)$, a regular model \mathcal{X}_U of X over U and an \mathbb{R} -Cartier divisor \mathcal{D}_U on \mathcal{X}_U such that g_P is given by \mathcal{D}_U for all $P \in U \cap M_K$. Moreover, we set

$$U' = \{P \in U \mid \mathcal{X}_U \rightarrow U \text{ is smooth over } P \text{ and } \mathcal{D}_U \text{ is horizontal over } P\}.$$

By Theorem 6.1.1, for each $P \in M_K$, we can find a Q -Green function q_P of $(C^0 \cap \text{PSH})$ -type on X_P^{an} such that (Q, q_P) yields the greatest element of

$$\left\{ (L, l_P) \left| \begin{array}{l} L \text{ is a nef } \mathbb{R}\text{-Cartier divisor on } X, l_P \text{ is an } L\text{-Green function} \\ \text{of } (C^0 \cap \text{PSH})\text{-type on } X_P^{\text{an}}, L \leq Q \text{ and } (L, l_P) \leq (D, g_P) \end{array} \right. \right\}.$$

For each $P \in U$, we set $(\mathcal{X}_U)_{(P)} := \mathcal{X}_U \times_U \text{Spec}((O_K)_P)$. Then, by Theorem 6.1.1 again, q_P is obtain by a relatively nef \mathbb{R} -Cartier divisor $\mathcal{Q}_{(P)}$ on $(\mathcal{X}_U)_{(P)}$. Note that if $P \in U'$, then $\mathcal{Q}_{(P)}$ is actually given by the Zariski closure of Q in $(\mathcal{X}_U)_{(P)}$. Therefore, we can find a relatively nef \mathbb{R} -Cartier divisor \mathcal{Q}_U on \mathcal{X}_U such that $\mathcal{Q}_U \cap (\mathcal{X}_U)_{(P)} = \mathcal{Q}_{(P)}$. Therefore, $\overline{Q} := (Q, \{q_P\}_{P \in M_K})$ forms a global adelic \mathbb{R} -Cartier divisor on X . By our construction, it is obvious that \overline{Q} is the greatest element of $\Sigma(\overline{D}; Q)$. Further, the second assertion of the theorem is also obvious by our construction. \square

As a corollary, we have the relative version of Corollary 6.2.7.

Corollary 6.2.2. *Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Let $\Upsilon_{\text{rel}}(\overline{D})$ be the set of all relatively nef adelic arithmetic \mathbb{R} -Cartier divisors \overline{L} of C^0 -type on X with $\overline{L} \leq \overline{D}$. If $\deg(D) \geq 0$, then there is the greatest element $\overline{Q} = (Q, q)$ of $\Upsilon_{\text{rel}}(\overline{D})$, that is, $\overline{Q} \in \Upsilon_{\text{rel}}(\overline{D})$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon_{\text{rel}}(\overline{D})$. Moreover, we have the following properties:*

- (1) $\overline{D} - \overline{Q}$ is vertical, that is, $D = Q$.
- (2) For every $a \in \mathbb{R}_{>0}$, the natural homomorphism $\hat{H}^0(X, a\overline{Q}^\tau) \rightarrow \hat{H}^0(X, a\overline{D}^\tau)$ is bijective. Further, $\|\phi\|_{aq_\infty} = \|\phi\|_{ag_\infty}$ for all $\phi \in H^0(X(\mathbb{C}), aD)$. In particular,

$$\hat{\chi}(X, a\overline{Q}) = \hat{\chi}(X, a\overline{D}) \quad \text{and} \quad \widehat{\text{vol}}_\chi(\overline{Q}) = \widehat{\text{vol}}_\chi(\overline{D}).$$

- (3) \overline{Q} is perpendicular to $\overline{D} - \overline{Q}$, that is, $\widehat{\deg}(\overline{Q} \cdot \overline{D} - \overline{Q}) = 0$.

Proof. Applying Theorem 6.2.1 to the case $Q = D$, we have the greatest element

$$\left(D, \sum_{P \in M_K} q_P[P] \right)$$

of $\Sigma(\overline{D}^\tau; D)$. Moreover, by using [26, Theorem 4.6], there is a D -Green function q_∞ of $(C^0 \cap \text{PSH})$ -type on $X(\mathbb{C})$ such that q_∞ yields the greatest element of

$$\{h_\infty \mid h_\infty \text{ is a } D\text{-Green function of } (C^0 \cap \text{PSH})\text{-type on } X(\mathbb{C}) \text{ and } h_\infty \leq g_\infty\}.$$

Thus $(D, \sum_{P \in M_K} q_P[P] + q_\infty[\infty])$ is our desired adelic arithmetic \mathbb{R} -Cartier divisor.

The property (1) is obvious. For (2), we suppose $\phi \in \hat{H}^0(X, a\overline{D}^\tau)$, that is,

$$-(1/a)(\phi)^a \leq \overline{D}^\tau$$

by Proposition 4.3.1. Note that $-(1/a)(\phi)^a$ is relatively nef, so that $-(1/a)(\phi)^a \leq \overline{Q}^\tau$. Therefore, $\phi \in \hat{H}^0(X, a\overline{Q}^\tau)$ by Proposition 4.3.1. The assertion $\|\cdot\|_{aq_\infty} = \|\cdot\|_{ag_\infty}$ on $H^0(X(\mathbb{C}), aD)$ follows from [28, Lemma 1.3]. Further, (3) is a consequence of (2) in Theorem 6.1.1 and [28, Lemma 1.3]. \square

The following theorem is one of the main results of this article.

Theorem 6.2.3. *Let \overline{D} be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Let R be an \mathbb{R} -Cartier divisor on X with $R \leq D$. Let $\Upsilon(\overline{D}; R)$ be the set of all nef adelic arithmetic \mathbb{R} -Cartier divisors $\overline{L} = (L, l)$ of C^0 -type on X with $L \leq R$ and $\overline{L} \leq \overline{D}$. If $\Upsilon(\overline{D}; R) \neq \emptyset$, then there is the greatest element \overline{Q} of $\Upsilon(\overline{D}; R)$, that is, $\overline{Q} \in \Upsilon(\overline{D}; R)$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon(\overline{D}; R)$.*

First let us consider two lemmas.

Lemma 6.2.4. *Let \overline{M} and \overline{Q} be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X and let $\{\overline{L}_n\}_{n=1}^\infty$ be a sequence of adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X with the following properties:*

- (1) \overline{L}_n is nef for all $n \geq 1$.
- (2) $\overline{M} \leq \overline{L}_n \leq \overline{Q}$ for all $n \geq 1$.
- (3) For all closed points x of X , $\lim_{n \rightarrow \infty} \text{mult}_x(L_n) = \text{mult}_x(Q)$.

Then $\widehat{\deg}(\overline{Q}|_x) \geq 0$ for all closed points $x \in X$.

Proof. We set $\overline{M} = (M, m)$, $\overline{Q} = (Q, q)$ and $\overline{L}_n = (L_n, g_n)$. First we assume that there are a regular model \mathcal{X} of X over $\text{Spec}(O_K)$ and an \mathbb{R} -Cartier divisor \mathcal{Q} on \mathcal{X} such that $\overline{Q} = (\mathcal{Q}, q_\infty)^a$. Let us see the following claim:

Claim 6.2.4.1. (i) *For all $n \geq 1$, there is an \mathbb{R} -Cartier divisor \mathcal{L}_n on \mathcal{X} such that $\mathcal{L}_n \cap X = L_n$, $(\mathcal{L}_n, (g_n)_\infty)^a$ is nef and $\overline{L}_n \leq (\mathcal{L}_n, (g_n)_\infty)^a \leq (\mathcal{Q}, q_\infty)^a$.*
(ii) *There is an \mathbb{R} -Cartier divisor \mathcal{M} on \mathcal{X} such that $\mathcal{M} \cap X = M$ and $(\mathcal{M}, m_\infty)^a \leq \overline{M}$.*

Proof. (i) For each $n \geq 1$, we consider the following set:

$$\{\overline{D} = (D, \{g_P\}_{P \in M_K}) \mid \overline{D} \text{ is relatively nef, } D \leq L_n \text{ and } \overline{D} \leq \mathcal{Q}^a\}.$$

Then, by using Theorem 6.2.1, there is a relatively nef \mathbb{R} -Cartier divisor \mathcal{L}_n on \mathcal{X} such that $(\mathcal{L}_n)^a$ gives rise to the greatest element of the above set. As \overline{L}_n belongs to the above set, we can see that $\overline{L}_n \leq (\mathcal{L}_n, (g_n)_\infty)^a$. Moreover, as $L_n = \mathcal{L}_n \cap X$, we have $(\mathcal{L}_n, (g_n)_\infty)^a$ is nef by Lemma 4.2.3, so that (i) follows.

(ii) There are an \mathbb{R} -Cartier divisor \mathcal{M}' on \mathcal{X} and a non-empty open set U of $\text{Spec}(O_K)$ such that m_P is defined by \mathcal{M}' for all $P \in U \cap M_K$. For each $P \in M_K \setminus U$, let m'_P be the M -Green function arising from \mathcal{M}' . As $m_P - m'_P$ is a continuous function on X_P , there is a constant θ_P such that $m_P \geq m'_P + \theta_P$ for all $P \in M_K \setminus U$. Let F_P be the fiber of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ over P . If we set

$$\overline{\mathcal{M}} = \left(\mathcal{M}' + \sum_{P \in M_K \setminus U} \frac{\theta_P}{2 \log \#(O_K/P)} F_P, m_\infty \right),$$

then $\overline{M} \geq \overline{\mathcal{M}}^a$, as required. \square

By the above claim, $(\mathcal{M}, m_\infty) \leq (\mathcal{L}_n, (g_n)_\infty) \leq (\mathcal{Q}, q_\infty)$ and $\mathcal{L}_n \cap X = L_n$ for all $n \geq 1$. Thus $\lim_{n \rightarrow \infty} \text{mult}_C(\mathcal{L}_n)$ exists for all prime divisors C on \mathcal{X} except finitely many fiber components, so that if we choose a subsequence $\{(\mathcal{L}_{n_i}, (g_{n_i})_\infty)\}_{i=1}^\infty$ of $\{(\mathcal{L}_n, (g_n)_\infty)\}_{n=1}^\infty$, then $\lim_{i \rightarrow \infty} \text{mult}_C(\mathcal{L}_{n_i})$ exists for all prime divisors C on \mathcal{X} . Therefore, by using [26, Theorem 7.1], there are an \mathbb{R} -Cartier divisor \mathcal{L} on \mathcal{X} and a Q -Green function g_∞ of $\text{PSH}_{\mathbb{R}}$ -type on $X(\mathbb{C})$ with the following properties:

- (a) $\text{mult}_C(\mathcal{L}) = \lim_{i \rightarrow \infty} \text{mult}_C(\mathcal{L}_{n_i})$ for all prime divisors C on \mathcal{X} . In particular, $\mathcal{L} \cap X = Q$.
- (b) $\widehat{\deg}((\mathcal{L}, g_\infty)|_C) \geq \limsup_{i \rightarrow \infty} \widehat{\deg}((\mathcal{L}_{n_i}, (g_{n_i})_\infty)|_C)$ for all prime divisors C on \mathcal{X} .
- (c) $(\mathcal{L}, g_\infty) \leq (\mathcal{Q}, q_\infty)$.

Let x be a closed point of X and Δ_x the closure of x in \mathcal{X} . Then, as $\mathcal{L} \cap X = \mathcal{Q} \cap X$ and $(\mathcal{L}, g_\infty) \leq (\mathcal{Q}, q_\infty)$,

$$\begin{aligned} \widehat{\deg}(\overline{Q}|_x) &= \widehat{\deg}((\mathcal{Q}, q_\infty)|_{\Delta_x}) \geq \widehat{\deg}((\mathcal{L}, g_\infty)|_{\Delta_x}) \\ &\geq \limsup_{i \rightarrow \infty} \widehat{\deg}((\mathcal{L}_{n_i}, (g_{n_i})_\infty)|_{\Delta_x}) \geq 0. \end{aligned}$$

In general, let U be a non-empty open set of $\text{Spec}(O_K)$ such that \bar{Q} has a defining model over U . For a positive number ϵ , by Theorem 4.1.3, there are a regular model \mathcal{X}_ϵ over $\text{Spec}(O_K)$ and an \mathbb{R} -Cartier divisor \mathcal{Q}_ϵ on \mathcal{X}_ϵ such that

$$\bar{Q} \leq (\mathcal{Q}_\epsilon, q_\infty)^a \leq \bar{Q} + \left(0, \sum_{P \in M_K \setminus U} \epsilon[P]\right).$$

Then, by the previous observation, $\widehat{\deg}((\mathcal{Q}_\epsilon, q_\infty)^a|_x) \geq 0$, so that, by Lemma 4.2.3,

$$\widehat{\deg} \left(\bar{Q} + \left(0, \sum_{P \in M_K \setminus U} \epsilon[P]\right) \Big|_x \right) \geq 0,$$

and hence

$$\widehat{\deg}(\bar{Q}|_x) \geq -\epsilon[K(x) : K] \#(M_K \setminus U),$$

where $K(x)$ is the residue field at x . Thus the assertion follows. \square

Lemma 6.2.5. (1) *Let*

$$\bar{L}_1 = (L_1, \{(l_1)_P\}_{P \in M_K}) \quad \text{and} \quad \bar{L}_2 = (L_2, \{(l_2)_P\}_{P \in M_K})$$

be global adelic \mathbb{R} -Cartier divisors of C^0 -type on X . If \bar{L}_1 and \bar{L}_2 are relatively nef, then

$$\max\{\bar{L}_1, \bar{L}_2\} := \left(\max\{L_1, L_2\}, \{\max\{(l_1)_P, (l_2)_P\}\}_{P \in M_K}\right)$$

is also relatively nef.

(2) *Let $\bar{Q}_1 = (Q_1, q_1)$ and $\bar{Q}_2 = (Q_2, q_2)$ be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . If \bar{Q}_1 and \bar{Q}_2 are nef, then*

$$\begin{aligned} \max\{\bar{Q}_1, \bar{Q}_2\} := & \left(\max\{Q_1, Q_2\}, \{\max\{(q_1)_P, (q_2)_P\}\}_{P \in M_K} \right. \\ & \left. \cup \{\max\{(q_1)_\infty, (q_2)_\infty\}\} \right) \end{aligned}$$

is also nef.

Proof. (1) We set $L_1 = a_{11}x_1 + \cdots + a_{1r}x_r$ and $L_2 = a_{21}x_1 + \cdots + a_{2r}x_r$, where x_1, \dots, x_r are closed points on X and $a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{2r} \in \mathbb{R}$. Let us choose an non-empty Zariski open set U of $\text{Spec}(O_K)$, a regular model \mathcal{X}_U over U and nef \mathbb{R} -Cartier divisors \mathcal{L}_1 and \mathcal{L}_2 on \mathcal{X}_U such that l_1 and l_2 are given by \mathcal{L}_1 and \mathcal{L}_2 over U , respectively. For $i = 1, \dots, r$, let C_i be the Zariski closure of x_i in \mathcal{X}_U . Shrinking U if necessarily, we may assume the following:

- (a) $\mathcal{L}_1 = a_{11}C_1 + \cdots + a_{1r}C_r$ and $\mathcal{L}_2 = a_{21}C_1 + \cdots + a_{2r}C_r$.
- (b) $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Then, by the properties (a) and (b), for $P \in U$, it is easy to see that $\max\{(l_1)_P, (l_2)_P\}$ is given by $g_{((\mathcal{X}_U)_{(P)}, \max\{\mathcal{L}_1, \mathcal{L}_2\}_{(P)})}$, where $(\mathcal{X}_U)_{(P)}$ is the localization of $\mathcal{X}_U \rightarrow U$ at P . Note that $\max\{\mathcal{L}_1, \mathcal{L}_2\}$ is relatively nef by (2) in Lemma 6.1.2. Moreover, for $P \in M_K \setminus U$, by (2) in Proposition 6.1.5, $\max\{(l_1)_P, (l_2)_P\}$ is of $(C^0 \cap \text{PSH})$ -type. Thus the assertion follows.

(2) By [26, Lemma 9.1.1], $\max\{(q_1)_\infty, (q_2)_\infty\}$ is of $(C^0 \cap \text{PSH})$ -type, so that, by virtue of (1), it is sufficient to show that $\widehat{\deg}(\max\{\bar{Q}_1, \bar{Q}_2\}|_x) \geq 0$ for all closed points x of X . As

$$\text{Supp}_{\mathbb{R}}(\max\{Q_1, Q_2\} - Q_1) \cap \text{Supp}_{\mathbb{R}}(\max\{Q_1, Q_2\} - Q_2) = \emptyset,$$

we may assume that $x \notin \text{Supp}_{\mathbb{R}}(\max\{Q_1, Q_2\} - Q_1)$. If we set

$$\bar{Q} := \max\{\bar{Q}_1, \bar{Q}_2\} - \bar{Q}_1,$$

then \overline{Q} is effective and $x \notin \text{Supp}_{\mathbb{R}}(Q)$, so that $\widehat{\deg}(\overline{Q}|_x) \geq 0$. Therefore,

$$\widehat{\deg}(\max\{\overline{Q}_1, \overline{Q}_2\}|_x) = \widehat{\deg}(\overline{Q}_1|_x) + \widehat{\deg}(\overline{Q}|_x) \geq 0,$$

as required. \square

Proof of Theorem 6.2.3. Let us start the proof of Theorem 6.2.3. We choose

$$\overline{M} = (M, m) \in \Upsilon(\overline{D}; R).$$

Let $\Upsilon([\overline{M}, \overline{D}]; R)$ be the set of all nef adelic arithmetic \mathbb{R} -Cartier divisors $\overline{L} = (L, l)$ of C^0 -type on X with $L \leq R$ and $\overline{M} \leq \overline{L} \leq \overline{D}$. For each closed point $x \in X$, we set

$$a_x = \sup \left\{ \text{mult}_x(L) \mid (L, l) \in \Upsilon([\overline{M}, \overline{D}]; R) \right\}.$$

Note that, if $x \notin \text{Supp}_{\mathbb{R}}(D) \cup \text{Supp}_{\mathbb{R}}(M)$, then $\text{mult}_x(L) = 0$ for all $\overline{L} = (L, l) \in \Upsilon([\overline{M}, \overline{D}]; R)$. In particular, $a_x = 0$, so that we set $Q = \sum_x a_x x$, which is an \mathbb{R} -Cartier divisor on X with $Q \leq R$. By using Theorem 6.2.1, let $(Q, \{q_P\}_{P \in M_K})$ be the greatest element of

$$A := \left\{ \overline{L} = (L, \{l_P\}_{P \in M_K}) \mid \begin{array}{l} \overline{L} \text{ is a relatively nef global adelic } \mathbb{R}\text{-Cartier} \\ \text{divisor on } X \text{ such that } L \leq Q \text{ and } \overline{L} \leq \overline{D}^\tau \end{array} \right\}.$$

Note that $(Q, \{q_P\}_{P \in M_K}) \geq \overline{M}^\tau$ because $\overline{M}^\tau \in A$. Moreover, by [26, Theorem 4.6], there is a Q -Green function q_∞ of $(C^0 \cap \text{PSH})$ -type on $X(\mathbb{C})$ such that q_∞ yields the greatest element of

$$B := \left\{ h_\infty \mid \begin{array}{l} h_\infty \text{ is a } Q\text{-Green function of } (C^0 \cap \text{PSH})\text{-type} \\ \text{on } X(\mathbb{C}) \text{ and } h_\infty \leq g_\infty \end{array} \right\}.$$

As $\max\{m_\infty, q_\infty\}$ is an element of B (cf. [26, Lemma 9.1.1]), we have $m_\infty \leq q_\infty$. Here we set

$$\overline{Q} := (Q, \{q_P\}_{P \in M_K} \cup \{q_\infty\}).$$

Clearly $\overline{M} \leq \overline{Q} \leq \overline{D}$ and $Q \leq R$.

Claim 6.2.6. *There is a sequence $\{\overline{L}_n = (L_n, l_n)\}_{n=1}^\infty$ in $\Upsilon([\overline{M}, \overline{D}]; R)$ such that*

$$a_x = \lim_{n \rightarrow \infty} \text{mult}_x(L_n)$$

for all closed points $x \in X$.

Proof. Let $\{x_1, \dots, x_N\} := \text{Supp}_{\mathbb{R}}(D) \cup \text{Supp}_{\mathbb{R}}(M)$. For each $i = 1, \dots, N$, we can find a sequence $\{\overline{L}_{i,n} = (L_{i,n}, l_{i,n})\}_{n=1}^\infty$ in $\Upsilon([\overline{M}, \overline{D}]; R)$ such that

$$a_{x_i} = \lim_{n \rightarrow \infty} \text{mult}_{x_i}(L_{i,n}).$$

We set $\overline{L}_n = \max\{\overline{L}_{1,n}, \dots, \overline{L}_{N,n}\}$ for $n \geq 1$. Then, by (2) in Lemma 6.2.5, $\overline{L}_n \in \Upsilon([\overline{M}, \overline{D}]; R)$, and hence

$$a_{x_i} = \lim_{n \rightarrow \infty} \text{mult}_{x_i}(L_n)$$

for $i = 1, \dots, N$. If $x \notin \{x_1, \dots, x_N\}$, then $a_x = 0$ and $\text{mult}_x(L_n) = 0$ for all n . Thus we have the claim. \square

As $\max\{q_\infty, (l_n)_\infty\} \in B$ by [26, Lemma 9.1.1], we have $(l_n)_\infty \leq q_\infty$. Moreover, $\overline{L}_n^\tau \leq \overline{Q}^\tau$ because $\overline{L}_n^\tau \in A$. Therefore, $\overline{L}_n \leq \overline{Q}$, so that, by Lemma 6.2.4, \overline{Q} is nef. In particular, $\overline{Q} \in \Upsilon(\overline{D}; R)$. We need to check that \overline{Q} is the greatest element of $\Upsilon(\overline{D}; R)$. Indeed, let $\overline{L} = (L, l) \in \Upsilon(\overline{D}; R)$. Then $\overline{L}^\tau \in A$ and $l_\infty \in B$, and hence $\overline{L}^\tau \leq \overline{Q}^\tau$ and $l_\infty \leq q_\infty$, as required. \square

Corollary 6.2.7 (Zariski decomposition for adelic arithmetic divisors). *Let \overline{D} be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Let $\Upsilon(\overline{D})$ be the set of all nef adelic arithmetic \mathbb{R} -Cartier divisors \overline{L} of C^0 -type on X with $\overline{L} \leq \overline{D}$. If $\Upsilon(\overline{D}) \neq \emptyset$, then there is the greatest element \overline{Q} of $\Upsilon(\overline{D})$, that is, $\overline{Q} \in \Upsilon(\overline{D})$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon(\overline{D})$. Moreover, the natural map $\hat{H}^0(X, a\overline{Q}) \rightarrow \hat{H}^0(X, a\overline{D})$ is bijective for $a \in \mathbb{R}_{>0}$. In particular, $\widehat{\text{vol}}(\overline{Q}) = \widehat{\text{vol}}(\overline{D})$.*

Proof. Applying Theorem 6.2.3 to the case where $R = D$, we have the first assertion. Let us see the second assertion. Let $\phi \in \hat{H}^0(X, a\overline{D})$, that is, $a\overline{D} + \widehat{(\phi)} \geq 0$ by Proposition 4.3.1, so that $\overline{D} \geq (1/a)\widehat{(\phi^{-1})}$. Note that $(1/a)\widehat{(\phi^{-1})}$ is nef, and hence $(1/a)\widehat{(\phi^{-1})} \in \Upsilon(\overline{D})$. Therefore, $\overline{Q} \geq (1/a)\widehat{(\phi^{-1})}$, that is, $a\overline{Q} + \widehat{(\phi)} \geq 0$, which means that $\phi \in \hat{H}^0(X, a\overline{Q})$ by Proposition 4.3.1. \square

Remark 6.2.8. There are several conditions to guarantee $\Upsilon(\overline{D}) \neq \emptyset$. For example, if $\hat{H}^0(X, a\overline{D}) \neq \{0\}$ for some $a \in \mathbb{R}_{>0}$, then $\Upsilon(\overline{D}) \neq \emptyset$. Indeed, $\hat{H}^0(X, a\overline{D}) \neq \{0\}$ implies that $a\overline{D} + \widehat{(\phi)} \geq 0$ for some $\phi \in \text{Rat}(X)^\times$, so that $(1/a)\widehat{(\phi^{-1})} \leq \overline{D}$. Note that $(1/a)\widehat{(\phi^{-1})}$ is nef, and hence $(1/a)\widehat{(\phi^{-1})} \in \Upsilon(\overline{D})$. In particular, if \overline{D} is big, then $\Upsilon(\overline{D}) \neq \emptyset$. As a conjecture, we expect that if \overline{D} is pseudo-effective, then $\Upsilon(\overline{D}) \neq \emptyset$.

7. CHARACTERIZATION OF NEF ADELIC ARITHMETIC DIVISORS ON ARITHMETIC SURFACES

In this section, we consider a generalization of the numerical characterization of nef arithmetic divisors proved in [28] to adelic arithmetic divisors. Namely, we will prove that an integrable adelic arithmetic \mathbb{R} -Cartier divisor \overline{D} of C^0 -type on a projective smooth curve over a number field is nef if and only if \overline{D} is pseudo-effective and $\widehat{\text{deg}}(\overline{D}^2) = \widehat{\text{vol}}(\overline{D})$. Throughout this section, let X be a projective, smooth and geometrically integral variety over a number field K .

7.1. Hodge index theorem for adelic arithmetic divisors. We assume that $\dim X = 1$. Let us start with a refinement of the generalized Hodge index theorem on an arithmetic surface.

Theorem 7.1.1. *Let $\overline{D} = (D, g)$ be an integrable adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . If $\text{deg}(D) \geq 0$, then $\widehat{\text{deg}}(\overline{D}^2) \leq \widehat{\text{vol}}_\chi(\overline{D})$. Moreover, the equality holds if and only if \overline{D} is relatively nef.*

Proof. Let $\Upsilon_{\text{rel}}(\overline{D})$ be the set of all relatively nef adelic arithmetic \mathbb{R} -Cartier divisors \overline{L} of C^0 -type on X with $\overline{L} \leq \overline{D}$ (cf. Corollary 6.2.2). Then, by Corollary 6.2.2, $\Upsilon_{\text{rel}}(\overline{D})$ has the greatest element $\overline{Q} = (Q, q)$, that is, $\overline{Q} \in \Upsilon_{\text{rel}}(\overline{D})$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon_{\text{rel}}(\overline{D})$. Further, we have the following:

- (1) $\widehat{\text{vol}}_\chi(\overline{Q}) = \widehat{\text{vol}}_\chi(\overline{D})$.
- (2) If we set $\overline{N} := \overline{D} - \overline{Q}$, then $\overline{N} = (0, \phi)$ and $\widehat{\text{deg}}(\overline{Q} \cdot \overline{N}) = 0$, where $\phi = \{\phi_P\}_{P \in M_K} \cup \{\phi_\infty\}$ is a collection of integrable continuous functions.

By Theorem 5.3.2,

$$\widehat{\text{deg}}(\overline{Q}^2) = \widehat{\text{vol}}_\chi(\overline{Q}) = \widehat{\text{vol}}_\chi(\overline{D}).$$

Note that

$$\widehat{\text{deg}}(\overline{N}^2) = \sum_{P \in M_K} \log \#(O_K/P) \widehat{\text{deg}}_P((0, \phi_P); \phi_P) + \frac{1}{2} \int_{X(\mathbb{C})} \phi_\infty dd^c(\phi_\infty),$$

so that, by Lemma 2.4.11 and [27, Proposition 1.2.3], we have

$$\widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\overline{Q}^2) + \widehat{\deg}(\overline{N}^2) \leq \widehat{\deg}(\overline{Q}^2).$$

Thus the first assertion follows. In addition, by using the equality conditions in Lemma 2.4.11 and [27, Proposition 1.2.3],

$$\begin{aligned} \widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}_{\chi}(\overline{D}) &\iff \deg(\overline{N}^2) = 0 \\ &\iff \phi_P \ (\forall P \in M_K) \text{ and } \phi_{\infty} \text{ are constant functions} \\ &\iff \overline{D} \text{ is relatively nef,} \end{aligned}$$

as required. \square

As a consequence of the above theorem, we have the Hodge index theorem for adelic arithmetic divisors.

Corollary 7.1.2. *Let $\overline{D} = (D, g)$ be an integrable adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . If $\deg(D) = 0$, then $\deg(\overline{D}^2) \leq 0$. Moreover, the equality holds if and only if $\overline{D} = (\psi)_{\mathbb{R}} + (0, \lambda[\infty])$ for some $\psi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ and $\lambda \in \mathbb{R}$.*

Proof. By Theorem 7.1.1, $\deg(\overline{D}^2) \leq \widehat{\text{vol}}_{\chi}(\overline{D}) \leq \widehat{\text{vol}}(\overline{D})$, so that it is sufficient to show that $\widehat{\text{vol}}(\overline{D}) = 0$ for the first assertion. Indeed, if $\widehat{\text{vol}}(\overline{D}) > 0$, then, by the continuity of the volume function (cf. Theorem 5.2.1), $\widehat{\text{vol}}(\overline{D} - (1/n)\overline{\mathcal{A}}) > 0$ for a sufficiently large n , where $\overline{\mathcal{A}}$ is an ample arithmetic Cartier divisor on some regular model of X . In particular, $\deg(D) \geq (1/n)\deg(\mathcal{A} \cap X) > 0$, which is a contradiction.

Next we consider the equality condition. Clearly if $\overline{D} = (\psi)_{\mathbb{R}} + (0, \lambda[\infty])$ for some $\psi \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ and $\lambda \in \mathbb{R}$, then $\widehat{\deg}(\overline{D}^2) = 0$, so that we assume that $\deg(\overline{D}^2) = 0$. Let \mathcal{X} be a regular model of X over $\text{Spec}(O_K)$.

Claim 7.1.2.1. *There is an \mathbb{R} -Cartier divisor \mathcal{D} on \mathcal{X} such that $\mathcal{D} \cap X = D$ and $(\mathcal{D} \cdot C) = 0$ for any vertical curves C on \mathcal{X} .*

Proof. Let U be a non-empty Zariski open set of $\text{Spec}(O_K)$ such that $\mathcal{X} \rightarrow \text{Spec}(O_K)$ is smooth over U . Let \mathcal{D}_0 be the Zariski closure of D in \mathcal{X} . Then the degree of \mathcal{D}_0 along every smooth fiber of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ is zero. Moreover, using Zariski's lemma, for each $P \in M_K \setminus U$, there is a vertical \mathbb{R} -Cartier divisor Z_P in the fiber F_P over P such that the degree of $\mathcal{D}_0 + Z_P$ along every irreducible component of F_P is zero. Therefore, if we set $\mathcal{D} = \mathcal{D}_0 + \sum_{P \in M_K \setminus U} Z_P$, we have our desired \mathbb{R} -Cartier divisor. \square

We set $\overline{D}' := (\mathcal{D}, g_{\infty})^a$, $\phi_P := g_P - g_{(\mathcal{X}_{(P)}, \mathcal{D}_{(P)})}$ and $\phi := \sum_{P \in M_K} \phi_P[P]$, where $\mathcal{X}_{(P)}$ is the localization of $\mathcal{X} \rightarrow \text{Spec}(O_K)$ at P and $\mathcal{D}_{(P)}$ is the restriction of \mathcal{D} to $\mathcal{X}_{(P)}$. Then $\overline{D} = \overline{D}' + (0, \phi)$. Note that $\widehat{\deg}(\overline{D}' \cdot (0, \phi)) = 0$ because $(\mathcal{D} \cdot C) = 0$ for any vertical curve C in \mathcal{X} , so that

$$0 = \widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\overline{D}'^2) + \widehat{\deg}((0, \phi)^2).$$

Therefore,

$$\widehat{\deg}(\overline{D}'^2) = \widehat{\deg}((0, \phi)^2) = 0$$

because $\widehat{\deg}(\overline{D}'^2) \leq 0$ and $\widehat{\deg}((0, \phi)^2) \leq 0$ by the first assertion. As

$$\widehat{\deg}(\overline{D}'^2) = \widehat{\deg}((\mathcal{D}, g_{\infty})^2) = 0,$$

by virtue of [28, Lemma 4.1],

$$\overline{D}' = (\widehat{\psi'})_{\mathbb{R}} + (0, \eta'[\infty])$$

for some $\psi' \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ and an F_{∞} -invariant locally constant function η' on $X(\mathbb{C})$. Moreover, by using the fact $\widehat{\deg}((0, \phi)^2) = 0$ together with Lemma 2.4.11, ϕ_P is a constant for all $P \in M_K$. Note that $\phi_P = 0$ expect finitely many $P \in M_K$. In addition, for each $P \in M_K$, there is $f_P \in K^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ such that $(f_P) = P$ on $\text{Spec}(O_K)$. Therefore,

$$\overline{D} = (\widehat{\psi''})_{\mathbb{R}} + (0, \eta''[\infty])$$

for some $\psi'' \in \text{Rat}(X)_{\mathbb{R}}^{\times}$ and an F_{∞} -invariant locally constant function η'' on $X(\mathbb{C})$. Thus, by using Dirichlet's unit theorem, we have the second assertion of the corollary (cf. [28, Proof of Lemma 4.1]). \square

7.2. Arithmetic asymptotic multiplicity. Let \mathbb{K} be either \mathbb{Q} or \mathbb{R} . Let

$$\text{Rat}(X)_{\mathbb{K}}^{\times} := \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{K},$$

and let

$$(\cdot)_{\mathbb{K}} : \text{Rat}(X)_{\mathbb{K}}^{\times} \rightarrow \text{Div}(X)_{\mathbb{K}} \quad \text{and} \quad \widehat{(\cdot)}_{\mathbb{K}} : \text{Rat}(X)_{\mathbb{K}}^{\times} \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$$

be the natural extensions of the homomorphisms

$$\text{Rat}(X)^{\times} \rightarrow \text{Div}(X) \quad \text{and} \quad \text{Rat}(X)^{\times} \rightarrow \widehat{\text{Div}}_{C^0}^a(X)_{\mathbb{R}}$$

given by $\phi \mapsto (\phi)$ and $\phi \mapsto \widehat{(\phi)}$, respectively. Let \overline{D} be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type. We define $\widehat{\Gamma}_{\mathbb{K}}^{\times}(X, \overline{D})$ to be

$$\widehat{\Gamma}_{\mathbb{K}}^{\times}(X, \overline{D}) := \left\{ \phi \in \text{Rat}(X)_{\mathbb{K}}^{\times} \mid \overline{D} + \widehat{(\phi)}_{\mathbb{K}} \geq (0, 0) \right\}.$$

For $\xi \in X$, the \mathbb{K} -asymptotic multiplicity of \overline{D} at ξ is defined to be

$$\mu_{\mathbb{K}, \xi}(\overline{D}) := \begin{cases} \inf \{ \text{mult}_{\xi}(D + (\phi)_{\mathbb{K}}) \mid \phi \in \widehat{\Gamma}_{\mathbb{K}}^{\times}(X, \overline{D}) \} & \text{if } \widehat{\Gamma}_{\mathbb{K}}^{\times}(X, \overline{D}) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 7.2.1. *Let \overline{D} and \overline{E} be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Then we have the following:*

- (1) $\mu_{\mathbb{K}, \xi}(\overline{D} + \overline{E}) \leq \mu_{\mathbb{K}, \xi}(\overline{D}) + \mu_{\mathbb{K}, \xi}(\overline{E})$.
- (2) If $\overline{D} \leq \overline{E}$, then $\mu_{\mathbb{K}, \xi}(\overline{E}) \leq \mu_{\mathbb{K}, \xi}(\overline{D}) + \text{mult}_{\xi}(E - D)$.
- (3) $\mu_{\mathbb{K}, \xi}(\overline{D} + \widehat{(\phi)}_{\mathbb{K}}) = \mu_{\mathbb{K}, \xi}(\overline{D})$ for $\phi \in \text{Rat}(X)_{\mathbb{K}}^{\times}$.
- (4) $\mu_{\mathbb{K}, \xi}(a\overline{D}) = a\mu_{\mathbb{K}, \xi}(\overline{D})$ for $a \in \mathbb{K}_{\geq 0}$.
- (5) $0 \leq \mu_{\mathbb{R}, \xi}(\overline{D}) \leq \mu_{\mathbb{Q}, \xi}(\overline{D})$.
- (6) If \overline{D} is big, then $\mu_{\mathbb{R}, \xi}(\overline{D}) = \mu_{\mathbb{Q}, \xi}(\overline{D})$.
- (7) If \overline{D} is nef and big, then $\mu_{\mathbb{K}, \xi}(\overline{D}) = 0$.

Proof. (1), (2), (3), (4) and (5) can be proved in the same way as [25, Proposition 2.1]. For the proofs for (6) and (7), let us begin with the following claim:

Claim 7.2.1.1. *Let $\overline{D}_1, \dots, \overline{D}_r$ be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . If \overline{D} is big, then*

$$\lim_{\substack{(x_1, \dots, x_r) \rightarrow (0, \dots, 0) \\ (x_1, \dots, x_r) \in \mathbb{Q}^r}} \mu_{\mathbb{Q}, \xi}(\overline{D} + x_1 \overline{D}_1 + \dots + x_r \overline{D}_r) = \mu_{\mathbb{Q}, \xi}(\overline{D}).$$

Proof. Here we define $f : \mathbb{Q}^r \rightarrow \mathbb{R} \cup \{\infty\}$ to be $f(x) := \mu_{\mathbb{Q}, \xi}(\overline{D}_x)$ for $x = (x_1, \dots, x_r) \in \mathbb{Q}^r$, where

$$\overline{D}_x = \overline{D} + x_1 \overline{D}_1 + \dots + x_r \overline{D}_r.$$

Note that f is a convex function over \mathbb{Q} , that is,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in \mathbb{Q}^r$ and $t \in [0, 1] \cap \mathbb{Q}$. Indeed, by using (1) and (4),

$$\begin{aligned} f(tx + (1-t)y) &= \mu_{\mathbb{Q}, \xi}(t\overline{D}_x + (1-t)\overline{D}_y) \leq \mu_{\mathbb{Q}, \xi}(t\overline{D}_x) + \mu_{\mathbb{Q}, \xi}((1-t)\overline{D}_y) \\ &= tf(x) + (1-t)f(y). \end{aligned}$$

Moreover, by virtue of the continuity of the volume function, there is a positive rational number c such that \overline{D}_x is big for all $x \in (-c, \infty)^r \cap \mathbb{Q}^r$. Therefore, by [24, Proposition 1.3.1], there is a continuous function $\tilde{f} : (-c, \infty)^r \rightarrow \mathbb{R}$ such that $\tilde{f} = f$ on $(-c, \infty)^r \cap \mathbb{Q}^r$. Thus the assertion follows. \square

(6) Let $\phi \in \widehat{\Gamma}_{\mathbb{R}}^{\times}(X, \overline{D})$, that is, $\phi = \phi_1^{a_1} \dots \phi_r^{a_r}$ and $\overline{D} + a_1(\widehat{\phi_1}) + \dots + a_r(\widehat{\phi_r}) \geq 0$ for some $\phi_1, \dots, \phi_r \in \text{Rat}(X)^{\times}$ and $a_1, \dots, a_r \in \mathbb{R}$. By using [26, Lemma 5.2.3 and Lemma 5.2.4], for each i , we can find effective adelic arithmetic \mathbb{R} -Cartier divisors \overline{A}_i and \overline{B}_i of C^0 -type on X such that $(\widehat{\phi_i}) = \overline{A}_i - \overline{B}_i$. For $n \in \mathbb{Z}_{>0}$, we choose $x_{i,n} \in \mathbb{R}$ and $x'_{i,n} \in \mathbb{Q}$ such that $a_i + x_{i,n} \in \mathbb{Q}$ and $0 \leq x_{i,n} \leq x'_{i,n} \leq 1/n$. Then

$$\begin{aligned} \overline{D} + \sum_i x'_{i,n} \overline{B}_i + \sum_i (a_i + x_{i,n})(\widehat{\phi_i}) &\geq \overline{D} + \sum_i x_{i,n} \overline{B}_i + \sum_i (a_i + x_{i,n})(\widehat{\phi_i}) \\ &= \overline{D} + \sum_i a_i(\widehat{\phi_i}) + \sum_i x_{i,n} \overline{A}_i \geq 0, \end{aligned}$$

and hence

$$\mu_{\mathbb{Q}, \xi} \left(\overline{D} + \sum_i x'_{i,n} \overline{B}_i \right) \leq \text{mult}_{\xi} \left(D + \sum_i x'_{i,n} B_i + \sum_i (a_i + x_{i,n})(\phi_i) \right).$$

On the other hand, by using Claim 7.2.1.1,

$$\lim_{n \rightarrow \infty} \mu_{\mathbb{Q}, \xi} \left(\overline{D} + \sum_i x'_{i,n} \overline{B}_i \right) = \mu_{\mathbb{Q}, \xi}(\overline{D})$$

and

$$\lim_{n \rightarrow \infty} \text{mult}_{\xi} \left(D + \sum_i x'_{i,n} B_i + \sum_i (a_i + x_{i,n})(\phi_i) \right) = \text{mult}_{\xi} \left(D + \sum_i a_i(\phi_i) \right).$$

Thus

$$\mu_{\mathbb{Q}, \xi}(\overline{D}) \leq \text{mult}_{\xi} \left(D + \sum_i a_i(\phi_i) \right),$$

which yields $\mu_{\mathbb{Q}, \xi}(\overline{D}) \leq \mu_{\mathbb{R}, \xi}(\overline{D})$, so that (6) follows from (5).

(7) By (5), it is sufficient to see that $\mu_{\mathbb{Q}, \xi}(\overline{D}) = 0$. We set

$$\overline{D} = (D, \{g_P\}_{P \in M_K} \cup \{g_{\infty}\}).$$

Let U be a non-empty open set $\text{Spec}(O_K)$ such that \overline{D} has a defining model over U . For each $n \in \mathbb{Z}_{>0}$, by Proposition 4.4.2, there is a normal model \mathcal{X} of X and a relatively nef \mathbb{R} -Cartier divisor \mathcal{D} on \mathcal{X} such that $\mathcal{D} \cap X = D$ and

$$(\mathcal{D}, g_{\infty})^a - (1/n) \left(0, \sum_{P \in M_K \setminus U} [P] \right) \leq \overline{D} \leq (\mathcal{D}, g_{\infty})^a.$$

Note that (\mathcal{D}, g_∞) is nef and big, so that, by (2) and [25, Proposition 2.1, (6)],

$$0 \leq \mu_{\mathbb{Q}, \xi} \left(\overline{D} + (1/n) \left(0, \sum_{P \in M_K \setminus U} [P] \right) \right) \leq \mu_{\mathbb{Q}, \xi}((\mathcal{D}, g_\infty)) = 0,$$

and hence $\mu_{\mathbb{Q}, \xi} \left(\overline{D} + (1/n) \left(0, \sum_{P \in M_K \setminus U} [P] \right) \right) = 0$. Further, by Claim 7.2.1.1,

$$\mu_{\mathbb{Q}, \xi}(\overline{D}) = \lim_{n \rightarrow \infty} \mu_{\mathbb{Q}, \xi} \left(\overline{D} + (1/n) \left(0, \sum_{P \in M_K \setminus U} [P] \right) \right),$$

so that (7) follows. \square

7.3. Necessary condition for the equality $\widehat{\text{vol}} = \widehat{\text{vol}}_\chi$. Let \overline{K} be an algebraic closure of K and $X_{\overline{K}} := X \times_{\text{Spec}(K)} \text{Spec}(\overline{K})$.

We fix a monomial order \preceq on $\mathbb{Z}_{\geq 0}^d$, that is, \preceq is a total ordering relation on $\mathbb{Z}_{\geq 0}^d$ with the following properties:

- (a) $(0, \dots, 0) \preceq A$ for all $A \in \mathbb{Z}_{\geq 0}^d$.
- (b) If $A \preceq B$ for $A, B \in \mathbb{Z}_{\geq 0}^d$, then $A + C \preceq B + C$ for all $C \in \mathbb{Z}_{\geq 0}^d$.

The monomial order \preceq on $\mathbb{Z}_{\geq 0}^d$ extends uniquely to a totally ordering relation \preceq on \mathbb{Z}^d such that $A + C \preceq B + C$ holds for all $A, B, C \in \mathbb{Z}^d$ with $A \preceq B$.

Let $z_P = (z_1, \dots, z_d)$ be a local system of parameters of $\mathcal{O}_{X_{\overline{K}}, P}$ at $P \in X(\overline{K})$. Note that the completion $\widehat{\mathcal{O}}_{X_{\overline{K}}, P}$ of $\mathcal{O}_{X_{\overline{K}}, P}$ with respect to the maximal ideal of $\mathcal{O}_{X_{\overline{K}}, P}$ is naturally isomorphic to $\overline{K}[[z_1, \dots, z_d]]$. Thus, for $f \in \mathcal{O}_{X_{\overline{K}}, P}$, we can put

$$f = \sum_{(a_1, \dots, a_d) \in \mathbb{Z}_{\geq 0}^d} c_{(a_1, \dots, a_d)} z_1^{a_1} \cdots z_d^{a_d}, \quad (c_{(a_1, \dots, a_d)} \in \overline{K}).$$

We define $\text{ord}_{z_P}^{\preceq}(f)$ to be

$$\text{ord}_{z_P}^{\preceq}(f) := \begin{cases} \min_{\preceq} \{(a_1, \dots, a_d) \mid c_{(a_1, \dots, a_d)} \neq 0\} & \text{if } f \neq 0, \\ \infty & \text{otherwise,} \end{cases}$$

which gives rise to a rank d valuation, that is, the following properties are satisfied:

- (i) $\text{ord}_{z_P}^{\preceq}(fg) = \text{ord}_{z_P}^{\preceq}(f) + \text{ord}_{z_P}^{\preceq}(g)$ for $f, g \in \mathcal{O}_{X_{\overline{K}}, P}$.
- (ii) $\min \{ \text{ord}_{z_P}^{\preceq}(f), \text{ord}_{z_P}^{\preceq}(g) \} \preceq \text{ord}_{z_P}^{\preceq}(f+g)$ for $f, g \in \mathcal{O}_{X_{\overline{K}}, P}$.

By the property (i), $\text{ord}_{z_P}^{\preceq} : \mathcal{O}_{X_{\overline{K}}, P} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}^d$ has the natural extension

$$\text{ord}_{z_P}^{\preceq} : \text{Rat}(X_{\overline{K}})^\times \rightarrow \mathbb{Z}^d$$

given by $\text{ord}_{z_P}^{\preceq}(f/g) = \text{ord}_{z_P}^{\preceq}(f) - \text{ord}_{z_P}^{\preceq}(g)$. Note that this extension also satisfies the same properties (i) and (ii) as before. Since $\text{ord}_{z_P}^{\preceq}(u) = (0, \dots, 0)$ for all $u \in \mathcal{O}_{X_{\overline{K}}, P}^\times$, $\text{ord}_{z_P}^{\preceq}$ induces $\text{Rat}(X_{\overline{K}})^\times / \mathcal{O}_{X_{\overline{K}}, P}^\times \rightarrow \mathbb{Z}^d$. The composition of homomorphisms

$$\text{Div}(X_{\overline{K}}) \xrightarrow{\alpha_P} \text{Rat}^\times(X_{\overline{K}}) / \mathcal{O}_{X_{\overline{K}}, P}^\times \xrightarrow{\text{ord}_{z_P}^{\preceq}} \mathbb{Z}^d$$

is denoted by $\text{mult}_{z_P}^{\preceq}$, where $\alpha_P : \text{Div}(X_{\overline{K}}) \rightarrow \text{Rat}(X_{\overline{K}})^\times / \mathcal{O}_{X_{\overline{K}}, P}^\times$ is the natural homomorphism. Moreover, the homomorphism $\text{mult}_{z_P}^{\preceq} : \text{Div}(X_{\overline{K}}) \rightarrow \mathbb{Z}^d$ gives rise to the natural extension $\text{Div}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathbb{R}^d$ over \mathbb{R} . By abuse of notation, the above extension is also denoted by $\text{mult}_{z_P}^{\preceq}$.

Let $\overline{D} = (D, g)$ be an adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . Let $V_\bullet = \bigoplus_{m \geq 0} V_m$ be a graded subalgebra of $R(D) := \bigoplus_{m \geq 0} H^0(X, mD)$ over K . The Okounkov body $\Delta(V_\bullet)$ of V_\bullet is defined by the closed convex hull of

$$\bigcup_{m > 0} \left\{ \text{mult}_{z_P}^{\leq} (D_{\overline{K}} + (1/m)(\phi)) \in \mathbb{R}_{\geq 0}^d \mid \phi \in (V_m \otimes_K \overline{K}) \setminus \{0\} \right\}.$$

For $t \in \mathbb{R}$, let V_\bullet^t be a graded subalgebra of V_\bullet given by

$$V_\bullet^t := \bigoplus_{m \geq 0} \left\langle V_m \cap \hat{H}^0(X, m(\overline{D} + (0, -2t[\infty]))) \right\rangle_K,$$

where $\left\langle V_m \cap \hat{H}^0(X, m(\overline{D} + (0, -2t[\infty]))) \right\rangle_K$ means the subspace of V_m generated by

$$V_m \cap \hat{H}^0(X, m(\overline{D} + (0, -2t[\infty])))$$

over K . We define $G_{(\overline{D}; V_\bullet)} : \Delta(V_\bullet) \rightarrow \mathbb{R} \cup \{-\infty\}$ to be

$$G_{(\overline{D}; V_\bullet)}(x) := \begin{cases} \sup \{t \in \mathbb{R} \mid x \in \Delta(V_\bullet^t)\} & \text{if } x \in \Delta(V_\bullet^t) \text{ for some } t, \\ -\infty & \text{otherwise.} \end{cases}$$

Note that $G_{(\overline{D}; V_\bullet)}$ is an upper semicontinuous concave function (cf. [5, SubSection 1.3]).

We also define $\widehat{\text{vol}}(\overline{D}; V_\bullet)$ and $\widehat{\text{vol}}_\chi(\overline{D}; V_\bullet)$ to be

$$\begin{cases} \widehat{\text{vol}}(\overline{D}; V_\bullet) := \limsup_{m \rightarrow \infty} \frac{\# \log(V_m \cap \hat{H}^0(X, m\overline{D}))}{m^{d+1}/(d+1)!}, \\ \widehat{\text{vol}}_\chi(\overline{D}; V_\bullet) := \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(V_m \cap H^0(X, mD), \|\cdot\|_{m\overline{D}})}{m^{d+1}/(d+1)!}. \end{cases}$$

Let $\Theta(\overline{D}; V_\bullet)$ be the closure of

$$\{x \in \Delta(V_\bullet) \mid G_{(\overline{D}; V_\bullet)}(x) > 0\}.$$

We assume that V_\bullet contains an ample series, that is, $V_m \neq \{0\}$ for $m \gg 1$ and there is an ample \mathbb{Q} -Cartier divisor A on X with the following properties:

- $A \leq D$.
- There is a positive integer m_0 such that $H^0(X, m m_0 A) \subseteq V_{m m_0}$ for all $m \geq 1$.

Then, in the similar way as [5, Theorem 2.8], [5, Theorem 3.1] and [28, Section 3], we have the following integral formulae:

$$(7.3.1) \quad \widehat{\text{vol}}(\overline{D}; V_\bullet) = (d+1)! [K : \mathbb{Q}] \int_{\Theta(\overline{D}; V_\bullet)} G_{(\overline{D}; V_\bullet)}(x) dx$$

and

$$(7.3.2) \quad \widehat{\text{vol}}_\chi(\overline{D}; V_\bullet) = (d+1)! [K : \mathbb{Q}] \int_{\Delta(V_\bullet)} G_{(\overline{D}; V_\bullet)}(x) dx.$$

Therefore, in the same way as [28, Section 3], we have the following theorem (cf. [28, Theorem 3.4 and Corollary 3.5]).

Theorem 7.3.3. *We assume that D is nef and big and $\widehat{\text{vol}}(\overline{D}) = \widehat{\text{vol}}_\chi(\overline{D}) > 0$. Then $\mu_{\mathbb{Q}, \xi}(\overline{D}) = 0$ for all $\xi \in X$.*

Besides it, the following theorem is also obtained:

Theorem 7.3.4. *Let $\bar{Q} = (Q, q)$ be a nef adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type on X . We assume that \bar{D} is big, $\bar{Q} \leq \bar{D}$ and $\widehat{\text{vol}}(\bar{Q}) = \widehat{\text{vol}}(\bar{D})$. If we set $N = D - Q$, then $\mu_{\mathbb{Q}, \xi}(\bar{D}) = \text{mult}_{\xi}(N)$ for all $\xi \in X$.*

Proof. By using (2) and (7) in Proposition 7.2.1, we have

$$\mu_{\mathbb{Q}, \xi}(\bar{D}) \leq \mu_{\mathbb{Q}, \xi}(\bar{Q}) + \text{mult}_{\xi}(N) = \text{mult}_{\xi}(N).$$

Let us consider the converse inequality. Let B be the Zariski closure of $\{\xi\}$ and P a regular closed point of B . Let $z_P = (z_1, \dots, z_d)$ be a local system of parameters of $\mathcal{O}_{X, P}$ such that B is given by $z_1 = \dots = z_r = 0$. We choose a monomial order \prec of $\mathbb{Z}_{\geq 0}^d$ such that $\ell(a) \leq \ell(b)$ for all $a, b \in \mathbb{Z}_{\geq 0}^d$ with $a \prec b$, where $\ell(x_1, \dots, x_d) = x_1 + \dots + x_d$. We set $v := \text{mult}_{z_P}^{\prec}(N)$. For simplicity, in the case $V_{\bullet} = R(D)$, we denote $\Delta(V_{\bullet})$, $\Delta(V_{\bullet}^t)$, $G_{(\bar{D}; V_{\bullet})}$ and $\Theta(\bar{D}; V_{\bullet})$ by Δ_D , Δ_D^t , $G_{\bar{D}}$ and $\Theta_{\bar{D}}$, respectively. Let us see the following claim:

Claim 7.3.4.1. (1) $\Delta_Q^t + v \subseteq \Delta_D^t$ for $t \in \mathbb{R}$.
 (2) $G_{\bar{Q}}(x) \leq G_{\bar{D}}(x + v)$ for $x \in \Delta_Q$.
 (3) $\Theta_{\bar{Q}} + v \subseteq \Theta_{\bar{D}}$.
 (4) $\min\{\ell(x) \mid x \in \Theta_{\bar{D}}\} \leq \mu_{\mathbb{Q}, \xi}(\bar{D})$.

Proof. (1) Let $\phi \in \langle \hat{H}^0(X, m(\bar{Q} + (0, -2t[\infty]))) \rangle_{\bar{K}} \setminus \{0\}$. Then

$$\text{mult}_{z_P}^{\prec}(Q + (1/m)(\phi)) + v = \text{mult}_{z_P}^{\prec}(D + (1/m)(\phi)),$$

which shows (1).

(2) Let t be a real number with $t < G_{\bar{Q}}(x)$. Then $x \in \Delta_Q^t \subseteq \Delta_D^t - v$ by (1), and hence $x + v \in \Delta_D^t$. Thus $t \leq G_{\bar{D}}(x + v)$, as required.

(3) follows because $G_{\bar{Q}}(x) > 0$ implies $G_{\bar{D}}(x + v) > 0$ by (2).

(4) Let $\phi \in \hat{H}^0(X, m\bar{D}) \setminus \{0\}$. Note that $\text{mult}_{z_P}^{\prec}(D + (1/m)(\phi)) \in \Theta_{\bar{D}}$ and

$$\ell(\text{mult}_{z_P}^{\prec}(D + (1/m)(\phi))) = \text{mult}_{\xi}(D + (1/m)(\phi)).$$

Thus $\min\{\ell(x) \mid x \in \Theta_{\bar{D}}\} \leq \text{mult}_{\xi}(D + (1/m)(\phi))$. Therefore, we have (4). \square

Since $\widehat{\text{vol}}(\bar{Q}) = \widehat{\text{vol}}(\bar{D})$, by using the integral formula (7.3.1) together with (2) and (3) in the above claim, we can see that $\Theta_{\bar{Q}} + v = \Theta_{\bar{D}}$. We choose $x_0 \in \Theta_{\bar{D}}$ such that $\ell(x_0) = \min\{\ell(x) \mid x \in \Theta_{\bar{D}}\}$. Then there is $y_0 \in \Theta_{\bar{Q}}$ such that $y_0 + v = x_0$. As $\ell(y_0) \geq 0$ and $\ell(v) = \text{mult}_{\xi}(N)$, by using (4) in the above claim,

$$\mu_{\mathbb{Q}, \xi}(\bar{D}) \geq \min\{\ell(x) \mid x \in \Theta_{\bar{D}}\} = \ell(x_0) = \ell(y_0) + \ell(v) \geq \ell(v) = \text{mult}_{\xi}(N),$$

as required. \square

Remark 7.3.5. By virtue of Theorem 7.3.4, we can generalize the necessary and sufficient condition for the existence of Zariski decompositions on arithmetic toric varieties proved in [4, Theorem 8.2] to the case of adelic arithmetic \mathbb{R} -divisors.

7.4. Numerical characterization. We assume that $\dim X = 1$. The following theorem is the main result of this section.

Theorem 7.4.1. *Let \bar{D} be an integrable adelic arithmetic \mathbb{R} -Cartier divisor on X . Then \bar{D} is nef if and only if \bar{D} is pseudo-effective and $\widehat{\deg}(\bar{D}^2) = \widehat{\text{vol}}(\bar{D})$.*

Proof. We need to show that if \overline{D} is pseudo-effective and $\widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}(\overline{D})$, then \overline{D} is nef because the converse follows from Proposition 4.4.2 and Theorem 7.1.1.

First we assume that \overline{D} is big. Since

$$\deg(\overline{D}^2) \leq \widehat{\text{vol}}_\chi(\overline{D}) \leq \widehat{\text{vol}}(\overline{D})$$

by Theorem 7.1.1, we have $\deg(\overline{D}^2) = \widehat{\text{vol}}_\chi(\overline{D})$ and $\widehat{\text{vol}}_\chi(\overline{D}) = \widehat{\text{vol}}(\overline{D})$. Thus, by Theorem 7.1.1 and Theorem 7.3.3, \overline{D} is relatively nef and $\mu_{\mathbb{Q},\xi}(\overline{D}) = 0$ for all $\xi \in X$. On the other hand, by Corollary 6.2.7, there is the greatest element \overline{Q} of $\Upsilon(\overline{D})$. Thus, if we set $\overline{N} := \overline{D} - \overline{Q}$, then $\text{mult}_\xi(N) = \mu_{\mathbb{Q},\xi}(\overline{D}) = 0$ for all $\xi \in X$ by Theorem 7.3.4, which means that $N = 0$. Therefore $\widehat{\deg}(\overline{D}|_x) \geq 0$ for all closed point $x \in X$, and hence \overline{D} is nef.

Next we suppose that $\deg(D) > 0$ and \overline{D} is not big. In this case, $\widehat{\deg}(\overline{D}^2) = \widehat{\text{vol}}(\overline{D}) = 0$. Thus, for $\epsilon > 0$, by using Proposition 5.1.2,

$$\epsilon[K : \mathbb{Q}] \deg(D) = \widehat{\deg}((\overline{D} + (0, \epsilon[\infty]))^2) \leq \widehat{\text{vol}}(\overline{D} + (0, \epsilon[\infty])) \leq \epsilon[K : \mathbb{Q}] \deg(D).$$

Therefore, $\widehat{\deg}((\overline{D} + (0, \epsilon[\infty]))^2) = \widehat{\text{vol}}(\overline{D} + (0, \epsilon[\infty])) > 0$, so that, by the previous observation, $\overline{D} + (0, \epsilon[\infty])$ is nef, and hence \overline{D} is also nef.

Finally we consider the case where $\deg(D) = 0$. Then $\widehat{\text{vol}}(\overline{D}) = 0$, so that $\widehat{\deg}(\overline{D}^2) = 0$. By Corollary 7.1.2, $\overline{D} = (\psi)_{\mathbb{R}} + (0, \lambda[\infty])$ for some $\psi \in \text{Rat}(X)_{\mathbb{R}}^\times$ and $\lambda \in \mathbb{R}$. As \overline{D} is pseudo-effective, by (2) in Proposition 4.5.4, $\widehat{\deg}(\overline{A} \cdot \overline{D}) \geq 0$ for any nef adelic arithmetic \mathbb{R} -Cartier divisor \overline{A} of C^0 -type. Thus we can see that $\lambda \geq 0$, and hence \overline{D} is nef. \square

Corollary 7.4.2. *Let \overline{D} and \overline{Q} be adelic arithmetic \mathbb{R} -Cartier divisors of C^0 -type on X . Then the following are equivalent:*

- (1) \overline{Q} is the greatest element of $\Upsilon(\overline{D})$, that is, $\overline{Q} \in \Upsilon(\overline{D})$ and $\overline{L} \leq \overline{Q}$ for all $\overline{L} \in \Upsilon(\overline{D})$.
- (2) \overline{Q} is an element of $\Upsilon(\overline{D})$ with the following property:

$$\widehat{\deg}(\overline{Q} \cdot \overline{B}) = 0 \quad \text{and} \quad \widehat{\deg}(\overline{B}^2) < 0$$

for all integrable adelic arithmetic \mathbb{R} -Cartier divisors \overline{B} of C^0 -type with $0 \not\leq \overline{B} \leq \overline{D} - \overline{Q}$.

Proof. (1) \implies (2) : By Corollary 6.2.7, $\widehat{\text{vol}}(\overline{D}) = \widehat{\text{vol}}(\overline{Q})$. Let \overline{B} be an integrable adelic arithmetic \mathbb{R} -Cartier divisor \overline{B} of C^0 -type with $0 \not\leq \overline{B} \leq \overline{D} - \overline{Q}$. For $0 < \epsilon \leq 1$,

$$\widehat{\deg}((\overline{Q} + \epsilon \overline{B})^2) \leq \widehat{\text{vol}}(\overline{Q} + \epsilon \overline{B})$$

by Theorem 7.1.1. On the other hand, by using Theorem 5.3.2,

$$\widehat{\deg}((\overline{Q} + \epsilon \overline{B})^2) \leq \widehat{\text{vol}}(\overline{Q} + \epsilon \overline{B}) \leq \widehat{\text{vol}}(\overline{D}) = \widehat{\text{vol}}(\overline{Q}) = \widehat{\deg}(\overline{Q}^2),$$

so that $\widehat{\deg}((\overline{Q} + \epsilon \overline{B})^2) \leq \widehat{\deg}(\overline{Q}^2)$. Therefore, $2\widehat{\deg}(\overline{Q} \cdot \overline{B}) + \epsilon\widehat{\deg}(\overline{B}^2) \leq 0$. In particular, $\widehat{\deg}(\overline{Q} \cdot \overline{B}) \leq 0$. Moreover, as \overline{Q} is nef and \overline{B} is effective, by (2) in Proposition 4.5.4, we have $\widehat{\deg}(\overline{Q} \cdot \overline{B}) \geq 0$, and hence $\widehat{\deg}(\overline{Q} \cdot \overline{B}) = 0$.

$\overline{Q} + \overline{B}$ is not nef because $\overline{B} \not\geq 0$, so that, by Theorem 7.4.1,

$$\widehat{\deg}((\overline{Q} + \overline{B})^2) < \widehat{\text{vol}}(\overline{Q} + \overline{B}) = \widehat{\text{vol}}(\overline{Q}) = \widehat{\deg}(\overline{Q}^2).$$

Therefore, $\widehat{\deg}(\overline{B}^2) < 0$.

(2) \implies (1) : Let \bar{L} be an element of $\Upsilon(\bar{D})$. If we set $\bar{A} := \max\{\bar{Q}, \bar{L}\}$ and $\bar{B} := \bar{A} - \bar{Q}$, then \bar{B} is effective, $\bar{A} \leq \bar{D}$ and \bar{A} is nef by Lemma 6.2.5. Moreover,

$$\bar{B} = \bar{A} - \bar{Q} \leq \bar{D} - \bar{Q}.$$

If we assume $\bar{B} \not\geq 0$, then, by the property, $\widehat{\deg}(\bar{Q} \cdot \bar{B}) = 0$ and $\widehat{\deg}(\bar{B}^2) < 0$. On the other hand, as \bar{A} is nef and \bar{B} is effective,

$$0 \leq \widehat{\deg}(\bar{A} \cdot \bar{B}) = \widehat{\deg}(\bar{Q} + \bar{B} \cdot \bar{B}) = \widehat{\deg}(\bar{B}^2),$$

which is a contradiction, so that $\bar{B} = 0$, that is, $\bar{Q} = \bar{A}$, which means that $\bar{L} \leq \bar{Q}$, as required. \square

Appendix A. CHARACTERIZATION OF RELATIVELY NEF CARTIER DIVISORS

In this appendix, we consider a characterization of relatively nef Cartier divisors in terms of asymptotic multiplicities. Let k be a field and ν a complete discrete valuation of k . Let ϖ be a uniformizing parameter of k° . Note that the valuation ν is not necessarily non-trivial.

A.1. Asymptotic multiplicity. Let \mathcal{X} be a $(d+1)$ -dimensional, proper and normal variety over k° (cf. Conventions and terminology 0.5.4), that is, the Krull dimension of \mathcal{X} is $d+1$, \mathcal{X} is proper over $\text{Spec}(k^\circ)$ and \mathcal{X} is integral and normal. We denote the rational function field of \mathcal{X} by $\text{Rat}(\mathcal{X})$. Let $\text{WDiv}(\mathcal{X})$ and $\text{Div}(\mathcal{X})$ denote the group of Weil divisors on \mathcal{X} and the group of Cartier divisors on \mathcal{X} , respectively. In addition, for a point $x \in \mathcal{X}$, let $\text{Div}(\mathcal{X}; x)$ be the subgroup of $\text{WDiv}(\mathcal{X})$ consisting of Weil divisors \mathcal{D} on \mathcal{X} such that $\mathcal{D} = (\phi)$ around x for some $\phi \in \text{Rat}(\mathcal{X})^\times$, that is, \mathcal{D} is a Cartier divisor around x . Note that

$$\text{Div}(\mathcal{X}) \subseteq \text{Div}(\mathcal{X}; x) \subseteq \text{WDiv}(\mathcal{X}).$$

For example, if x is a regular point of \mathcal{X} , then $\text{Div}(\mathcal{X}; x) = \text{WDiv}(\mathcal{X})$. We set

$$\begin{cases} \text{WDiv}(\mathcal{X})_{\mathbb{R}} := \text{WDiv}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R}, \\ \text{Div}(\mathcal{X})_{\mathbb{R}} := \text{Div}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R}, \\ \text{Div}(\mathcal{X}; x)_{\mathbb{R}} := \text{Div}(\mathcal{X}; x) \otimes_{\mathbb{Z}} \mathbb{R}. \end{cases}$$

Let \mathbb{K} be either \mathbb{Q} or \mathbb{R} and let $\text{Rat}(\mathcal{X})_{\mathbb{K}}^\times := (\text{Rat}(\mathcal{X})^\times, \times) \otimes_{\mathbb{Z}} \mathbb{K}$. Note that the homomorphism

$$(\cdot) : \text{Rat}(\mathcal{X})^\times \rightarrow \text{Div}(\mathcal{X})$$

given by $f \mapsto (f)$ has the natural extension

$$(\cdot) : \text{Rat}(\mathcal{X})_{\mathbb{K}}^\times \rightarrow \text{Div}(\mathcal{X})_{\mathbb{R}},$$

that is, for $\phi = f_1^{\otimes a_1} \cdots f_r^{\otimes a_r} \in \text{Rat}(\mathcal{X})_{\mathbb{K}}^\times$ ($f_1, \dots, f_r \in \text{Rat}(\mathcal{X})^\times$, $a_1, \dots, a_r \in \mathbb{K}$),

$$(\phi) = a_1(f_1) + \cdots + a_r(f_r).$$

Let \mathcal{D} be an \mathbb{R} -Weil divisor on \mathcal{X} , that is, $\mathcal{D} \in \text{WDiv}(\mathcal{X})_{\mathbb{R}}$. We define $\Gamma^\times(\mathcal{X}, \mathcal{D})$ and $\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D})$ to be

$$\begin{cases} \Gamma^\times(\mathcal{X}, \mathcal{D}) := \{\phi \in \text{Rat}(\mathcal{X})^\times \mid \mathcal{D} + (\phi) \geq 0\}, \\ \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D}) := \{\phi \in \text{Rat}(\mathcal{X})_{\mathbb{K}}^\times \mid \mathcal{D} + (\phi) \geq 0\}. \end{cases}$$

Let $w : \text{Rat}(\mathcal{X}) \rightarrow \mathbb{R} \cup \{\infty\}$ be an additive discrete valuation over k . Namely, w satisfies the following conditions:

- (1) $w(f \cdot g) = w(f) + w(g)$ for all $f, g \in \text{Rat}(\mathcal{X})$.
- (2) $w(f + g) \geq \min\{w(f), w(g)\}$ for all $f, g \in \text{Rat}(\mathcal{X})$.
- (3) $f = 0$ if and only if $w(f) = \infty$.
- (4) $w(a) = -\log v(a)$ for all $a \in k^\times$.

Let \mathcal{O}_w be the valuation ring of w and m_w its maximal ideal, that is,

$$\mathcal{O}_w = \{f \in \text{Rat}(\mathcal{X}) \mid w(f) \geq 0\} \quad \text{and} \quad m_w = \{f \in \text{Rat}(\mathcal{X}) \mid w(f) > 0\}.$$

Note that $k^\circ \subseteq \mathcal{O}_w$ and $k^\circ \subseteq m_w$, so that \mathcal{O}_w/m_w is a k°/k° -algebra. We say w is a *divisorial valuation of $\text{Rat}(\mathcal{X})$ over k* if $\text{trdeg}_{k^\circ/k^\circ} \mathcal{O}_w/m_w = d$. For a divisorial valuation w of $\text{Rat}(\mathcal{X})$ over k , there are a normal variety \mathcal{V} over k° , a vertical prime divisor Γ on \mathcal{V} and a birational morphism $\mu : \mathcal{V} \rightarrow \mathcal{X}$ over $\text{Spec}(k^\circ)$ such that $w = a \cdot \text{ord}_\Gamma$ for some $a \in \mathbb{R}_{>0}$. Indeed, it can be shown as follows: We may assume that v is non-trivial. Otherwise the assertion follows from [34, Chapter VI, § 14, Theorem 31]. We choose $x_1, \dots, x_d \in \mathcal{O}_w$ such that x_1, \dots, x_d form a transcendental basis of \mathcal{O}_w/m_w over k°/k° . Then $\text{Rat}(\mathcal{X})$ is a finite extension of $k(x_1, \dots, x_d)$ and the transcendental degree of $k^\circ[x_1, \dots, x_d]/k^\circ[x_1, \dots, x_d] \cap m_w$ over k°/k° is d . Let R be the normalization of $k^\circ[x_1, \dots, x_d]$ in $\text{Rat}(\mathcal{X})$. Note that R is finite over $k^\circ[x_1, \dots, x_d]$ because k° is excellent. In addition, $R \subseteq \mathcal{O}_w$, $R \cap m_w$ is a prime ideal of R and $\text{trdeg}_{k^\circ/k^\circ}(R/R \cap m_w) = d$, which prove the assertion.

We denote the set of all divisorial valuations of $\text{Rat}(\mathcal{X})$ over k by $\text{DVal}_k(\mathcal{X})$. As \mathcal{X} is proper and separated over $\text{Spec}(k^\circ)$, there is a unique morphism $t : \text{Spec}(\mathcal{O}_w) \rightarrow \mathcal{X}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \text{Spec}(\text{Rat}(\mathcal{X})) & \xrightarrow{\quad} & \mathcal{X} \\ \downarrow & \nearrow t & \downarrow \\ \text{Spec}(\mathcal{O}_w) & \xrightarrow{\quad} & \text{Spec}(k^\circ) \end{array}$$

Let x be the image of the closed point m_w by t . The point x is called the *center of w on \mathcal{X}* . Note that $x \in \mathcal{X}_\circ$ (the central fiber of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$). For $\mathcal{D} \in \text{Div}(\mathcal{X}; x)$, $\text{mult}_w(\mathcal{D})$ is defined by $w(f)$, where f is a local equation of \mathcal{D} at x . In this way, we have a map

$$\text{mult}_w : \text{Div}(\mathcal{X}; x) \rightarrow \mathbb{Z}.$$

It is easy to see that mult_w is a homomorphism, so that we have the natural extension

$$\text{mult}_w : \text{Div}(\mathcal{X}; x)_{\mathbb{R}} \rightarrow \mathbb{R},$$

that is,

$$\text{mult}_w(a_1 \mathcal{D}_1 + \dots + a_r \mathcal{D}_r) = a_1 \text{mult}_w(\mathcal{D}_1) + \dots + a_r \text{mult}_w(\mathcal{D}_r),$$

where $\mathcal{D}_1, \dots, \mathcal{D}_r \in \text{Div}(\mathcal{X}; x)$ and $a_1, \dots, a_r \in \mathbb{R}$.

For $\mathcal{D} \in \text{Div}(\mathcal{X}; x)_{\mathbb{R}}$, we define $\mu_{\mathbb{K}, w}(\mathcal{D})$ to be

$$\mu_{\mathbb{K}, w}(\mathcal{D}) := \begin{cases} \inf\{\text{mult}_w(\mathcal{D} + (\phi)) \mid \phi \in \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D})\} & \text{if } \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases}$$

which is called the *\mathbb{K} -asymptotic multiplicity of \mathcal{D} at w* . Here we give one additional definition. An \mathbb{R} -Cartier divisor \mathcal{D} (i.e. $\mathcal{D} \in \text{Div}(\mathcal{X})_{\mathbb{R}}$) is said to be *big* if \mathcal{D} is big on

the generic fiber $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$. First let us observe elementary properties of the asymptotic multiplicity. The arithmetic version can be found in [25, Proposition 2.1 and Theorem 2.5] and Proposition 7.2.1.

Proposition A.1.1. *Let w be a divisorial valuation of $\text{Rat}(\mathcal{X})$ over k and x the center of w on \mathcal{X} . For $\mathcal{D}, \mathcal{E} \in \text{Div}(\mathcal{X}; x)_{\mathbb{R}}$, we have the following:*

- (1) $\mu_{\mathbb{K}, w}(\mathcal{D} + \mathcal{E}) \leq \mu_{\mathbb{K}, w}(\mathcal{D}) + \mu_{\mathbb{K}, w}(\mathcal{E})$.
- (2) If $\mathcal{D} \leq \mathcal{E}$, then $\mu_{\mathbb{K}, w}(\mathcal{E}) \leq \mu_{\mathbb{K}, w}(\mathcal{D}) + \text{mult}_w(\mathcal{E} - \mathcal{D})$.
- (3) $\mu_{\mathbb{K}, w}(\mathcal{D} + (\phi)) = \mu_{\mathbb{K}, w}(\mathcal{D})$ for $\phi \in \text{Rat}(\mathcal{X})_{\mathbb{K}}^\times$.
- (4) $\mu_{\mathbb{K}, w}(a\mathcal{D}) = a\mu_{\mathbb{K}, w}(\mathcal{D})$ for $a \in \mathbb{K}_{>0}$.
- (5) $0 \leq \mu_{\mathbb{R}, w}(\mathcal{D}) \leq \mu_{\mathbb{Q}, w}(\mathcal{D})$.
- (6) Let $v : \mathcal{Y} \rightarrow \mathcal{X}$ be a birational morphism of proper and normal varieties over k° .
 - (6.1) If \mathcal{D} is an \mathbb{R} -Cartier divisor on \mathcal{X} , then $\mu_{\mathbb{K}, w}(v^*(\mathcal{D})) = \mu_{\mathbb{K}, w}(\mathcal{D})$.
 - (6.2) Let x and y be the centers of w on \mathcal{X} and \mathcal{Y} , respectively (note that $v(y) = x$). We assume that v is an isomorphism over x . Then, for $\mathcal{D}' \in \text{Div}(\mathcal{Y}; y)$,

$$\mu_{\mathbb{K}, w}(v_*(\mathcal{D}')) \leq \mu_{\mathbb{K}, w}(\mathcal{D}').$$

- (7) If \mathcal{D} is an \mathbb{R} -Cartier divisor on \mathcal{X} and \mathcal{D} is relatively nef with respect to $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ (cf. Conventions and terminology 0.5.6) and big, then $\mu_{\mathbb{K}, w}(\mathcal{D}) = 0$.
- (8) If \mathcal{D} is an \mathbb{R} -Cartier divisor on \mathcal{X} and \mathcal{D} is big, then $\mu_{\mathbb{Q}, w}(\mathcal{D}) = \mu_{\mathbb{R}, w}(\mathcal{D})$.

Proof. (1) If $\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D} + \mathcal{E}) = \emptyset$, then either $\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D}) = \emptyset$ or $\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{E}) = \emptyset$, so that we may assume that $\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D} + \mathcal{E}) \neq \emptyset$. Thus we may also assume that $\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset$ and $\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{E}) \neq \emptyset$. Therefore, the assertion follows because $\phi\psi \in \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D} + \mathcal{E})$ for all $\phi \in \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D})$ and $\psi \in \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{E})$.

(2) is derived from (1).

(3) The assertion follows from the following:

$$\psi \in \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D}) \iff \psi\phi^{-1} \in \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D} + (\phi)).$$

(4) Note that $\psi \in \Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D})$ if and only if $\psi^a \in \Gamma_{\mathbb{K}}^\times(\mathcal{X}, a\mathcal{D})$, and that

$$\text{mult}_w(a\mathcal{D} + (\psi^a)) = a \text{mult}_w(\mathcal{D} + (\psi)),$$

which implies (4).

(5) is obvious.

(6.1) For $\phi \in \text{Rat}(\mathcal{X})_{\mathbb{K}}^\times$, $\mathcal{D} + (\phi)_{\mathcal{X}} \geq 0$ if and only if $v^*(\mathcal{D}) + (\phi)_{\mathcal{Y}} \geq 0$. Thus

$$\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D}) = \Gamma_{\mathbb{K}}^\times(\mathcal{Y}, v^*(\mathcal{D})).$$

Moreover,

$$\text{mult}_w(\mathcal{D} + (\phi)_{\mathcal{X}}) = \text{mult}_w(v^*(\mathcal{D}) + (\phi)_{\mathcal{Y}}).$$

Therefore, we have (6.1).

(6.2) Let $\phi \in \Gamma_{\mathbb{K}}^\times(\mathcal{Y}, \mathcal{D}')$, that is, $\mathcal{D}' + (\phi)_{\mathcal{Y}} \geq 0$. Then

$$0 \leq v_*(\mathcal{D}' + (\phi)_{\mathcal{Y}}) = v_*(\mathcal{D}') + (\phi)_{\mathcal{X}}.$$

The above observation means that $\Gamma_{\mathbb{K}}^\times(\mathcal{Y}, \mathcal{D}') \subseteq \Gamma_{\mathbb{K}}^\times(\mathcal{X}, v_*(\mathcal{D}'))$. Moreover, by our assumption,

$$\text{mult}_w(\mathcal{D}' + (\phi)_{\mathcal{Y}}) = \text{mult}_w(v_*(\mathcal{D}') + (\phi)_{\mathcal{X}})$$

for $\phi \in \Gamma_{\mathbb{K}}^\times(\mathcal{Y}, \mathcal{D}')$. Thus the assertion follows.

(7) Let us begin with the following claim:

Claim A.1.1.1. *If \mathcal{X} is projective over $\mathrm{Spec}(k^\circ)$, then, for any Cartier divisor \mathcal{E} on \mathcal{X} , there are effective Cartier divisors \mathcal{A} and \mathcal{B} on \mathcal{X} such that $\mathcal{E} = \mathcal{A} - \mathcal{B}$.*

Proof. Let \mathcal{H} be an ample Cartier divisor on \mathcal{X} . Let $\mathcal{E} = e_1\Gamma_1 + \cdots + e_r\Gamma_r$ be the decomposition as a Weil divisor. As \mathcal{H} is ample, for a sufficiently large l , there is $\phi \in H^0(\mathcal{X}, l\mathcal{H}) \setminus \{0\}$ such that $\mathcal{A} := l\mathcal{H} + (\phi)$ is effective and $\mathrm{mult}_{\Gamma_i}(\mathcal{A}) \geq e_i$ for $i = 1, \dots, r$. Thus $\mathcal{B} := \mathcal{A} - \mathcal{E}$ is effective and $\mathcal{E} = \mathcal{A} - \mathcal{B}$. \square

Let us go back to the proof of (7). By (5), it is sufficient to show that $\mu_{\mathbb{Q},w}(\mathcal{D}) = 0$. By using Chow's lemma together with (6.1), we may assume that \mathcal{X} is projective over $\mathrm{Spec}(k^\circ)$. First we assume that \mathcal{D} is an ample \mathbb{Q} -Cartier divisor. Then there is $\phi \in \Gamma_{\mathbb{Q}}^\times(\mathcal{X}, \mathcal{D})$ such that $\mathrm{mult}_w(\mathcal{D} + (\phi)) = 0$, and hence $\mu_{\mathbb{Q},w}(\mathcal{D}) = 0$.

Next we assume that \mathcal{D} is ample. By using Claim A.1.1.1, we can set

$$\mathcal{D} = a_1\mathcal{D}_1 + \cdots + a_r\mathcal{D}_r,$$

where $\mathcal{D}_1, \dots, \mathcal{D}_r$ are effective Cartier divisors and $a_1, \dots, a_r \in \mathbb{R}$. For any $n > 0$, there are $\delta_1, \dots, \delta_r \in \mathbb{R}$ such that $0 < \delta_i < 1/n$ and $a_i - \delta_i \in \mathbb{Q}$ for all i and that $(a_1 - \delta_1)\mathcal{D}_1 + \cdots + (a_r - \delta_r)\mathcal{D}_r$ is ample. Then, by (2) and the previous case,

$$\begin{aligned} \mu_{\mathbb{Q},w}(\mathcal{D}) &\leq \mu_{\mathbb{Q},w}((a_1 - \delta_1)\mathcal{D}_1 + \cdots + (a_r - \delta_r)\mathcal{D}_r) + \mathrm{mult}_{\mathbb{Q},w}(\delta_1\mathcal{D}_1 + \cdots + \delta_r\mathcal{D}_r) \\ &\leq \delta_1 \mathrm{mult}_w(\mathcal{D}_1) + \cdots + \delta_r \mathrm{mult}_w(\mathcal{D}_r) \\ &\leq (1/n)(\mathrm{mult}_w(\mathcal{D}_1) + \cdots + \mathrm{mult}_w(\mathcal{D}_r)), \end{aligned}$$

which proves the assertion in this case.

Let us consider a general case.

Claim A.1.1.2. *There are an ample \mathbb{Q} -Cartier divisor \mathcal{A} on \mathcal{X} and $\phi \in \mathrm{Rat}(\mathcal{X})_{\mathbb{Q}}^\times$ such that $\mathcal{E} := \mathcal{D} - \mathcal{A} + (\phi)$ is effective.*

Proof. If v is trivial, then $k^\circ = k$, so that the assertion is obvious. We assume that v is non-trivial. Let \mathcal{A}' be an ample Cartier divisor on \mathcal{X} . Let X be the generic fiber of $\mathcal{X} \rightarrow \mathrm{Spec}(k^\circ)$, $D := \mathcal{D} \cap X$ and $A' := \mathcal{A}' \cap X$. Then, as D is big, there are $n \in \mathbb{Z}_{>0}$ and $\phi_1 \in \mathrm{Rat}(\mathcal{X})^\times$ such that

$$nD - A' + (\phi_1) \geq 0$$

on X . Therefore, we can find $m \in \mathbb{Z}_{>0}$ such that $n\mathcal{D} - \mathcal{A}' + (\phi_1) + m(\varpi) \geq 0$, and hence, $\mathcal{D} - (1/n)\mathcal{A}' + (\phi_1^{1/n}\varpi^{m/n}) \geq 0$, as required. \square

As $\mathcal{A} + (1 - \epsilon)\mathcal{E} = \epsilon\mathcal{A} + (1 - \epsilon)(\mathcal{D} + (\phi))$ is ample for $0 < \epsilon < 1$, by using (2), (3) and the previous assertion in the case where \mathcal{D} is ample,

$$\begin{aligned} \mu_{\mathbb{Q},w}(\mathcal{D}) &= \mu_{\mathbb{Q},w}(\mathcal{A} + \mathcal{E} + (\phi^{-1})) = \mu_{\mathbb{Q},w}(\mathcal{A} + \mathcal{E}) = \mu_{\mathbb{Q},w}(\mathcal{A} + (1 - \epsilon)\mathcal{E} + \epsilon\mathcal{E}) \\ &\leq \mu_{\mathbb{Q},w}(\mathcal{A} + (1 - \epsilon)\mathcal{E}) + \epsilon \mathrm{mult}_w(\mathcal{E}) \leq \epsilon \mathrm{mult}_w(\mathcal{E}), \end{aligned}$$

and hence $\mu_{\mathbb{Q},w}(\mathcal{D}) = 0$.

(8) In the same way as (7), we may assume that \mathcal{X} is projective over $\mathrm{Spec}(k^\circ)$. Let $\phi = \phi_1^{a_1} \cdots \phi_r^{a_r} \in \Gamma_{\mathbb{R}}^\times(\mathcal{X}, \mathcal{D})$, where $\phi_1, \dots, \phi_r \in \mathrm{Rat}(\mathcal{X})^\times$ and $a_1, \dots, a_r \in \mathbb{R}$. By Claim A.1.1.1, for each i , there are effective Cartier divisors \mathcal{A}_i and \mathcal{B}_i on \mathcal{X} such that $(\phi_i) = \mathcal{A}_i - \mathcal{B}_i$. Here we consider a map $f : \mathbb{Q}^r \rightarrow \mathbb{R} \cup \{\infty\}$ given by

$$f(t_1, \dots, t_r) = \mu_{\mathbb{Q},w}(\mathcal{D} + t_1\mathcal{B}_1 + \cdots + t_r\mathcal{B}_r).$$

Claim A.1.1.3. $\lim_{\substack{(t_1, \dots, t_r) \rightarrow (0, \dots, 0) \\ (t_1, \dots, t_r) \in \mathbb{Q}^r}} f(t_1, \dots, t_r) = f(0, \dots, 0) = \mu_{\mathbb{Q},w}(\mathcal{D}).$

Proof. First note that there is a positive rational number c such that $\mathcal{D} + t_1 \mathcal{B}_1 + \cdots + t_r \mathcal{B}_r$ is big for $(t_1, \dots, t_r) \in (-c, \infty)^r \cap \mathbb{Q}^r$. Moreover, by using (1) and (4), we can see that f is a convex function over \mathbb{Q} , that is, $f(\lambda t + (1 - \lambda)t') \leq \lambda f(t) + (1 - \lambda)f(t')$ for $t, t' \in \mathbb{Q}^r$ and $\lambda \in [0, 1] \cap \mathbb{Q}$. Therefore, by virtue of [24, Proposition 1.3.1], there is a continuous function $\tilde{f} : (-c, \infty)^r \rightarrow \mathbb{R}$ such that $\tilde{f} = f$ on $(-c, \infty)^r \cap \mathbb{Q}^r$, which shows the assertion of the claim. \square

For each $i = 1, \dots, r$ and $n \in \mathbb{Z}_{>0}$, we choose $t_{i,n} \in \mathbb{R}$ and $t'_{i,n} \in \mathbb{Q}$ such that $a_i + t_{i,n} \in \mathbb{Q}$ and $0 \leq t_{i,n} \leq t'_{i,n} \leq 1/n$. Then

$$\begin{aligned} \mathcal{D} + \sum_i t'_{i,n} \mathcal{B}_i + \sum_i (a_i + t_{i,n})(\phi_i) &\geq \mathcal{D} + \sum_i t_{i,n} \mathcal{B}_i + \sum_i (a_i + t_{i,n})(\phi_i) \\ &= \mathcal{D} + \sum_i a_i(\phi_i) + \sum_i t_{i,n} \mathcal{A}_i \geq 0, \end{aligned}$$

and hence

$$\mu_{\mathbb{Q},w} \left(\mathcal{D} + \sum_i t'_{i,n} \mathcal{B}_i \right) \leq \text{mult}_w \left(\mathcal{D} + \sum_i t'_{i,n} \mathcal{B}_i + \sum_i (a_i + t_{i,n})(\phi_i) \right).$$

Thus, taking the limits as $n \rightarrow \infty$ together with Claim A.1.1.3, we have

$$\mu_{\mathbb{Q},w}(\mathcal{D}) \leq \text{mult}_w \left(\mathcal{D} + \sum_i a_i(\phi_i) \right),$$

which gives rise to $\mu_{\mathbb{Q},w}(\mathcal{D}) \leq \mu_{\mathbb{R},w}(\mathcal{D})$, so that (8) follows from (5). \square

A.2. Sectional decomposition. Let \mathcal{X} be a regular and proper variety over k° . Let \mathcal{D} be an \mathbb{R} -Cartier divisor on \mathcal{X} . We assume that $\Gamma^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset$. We set

$$\begin{cases} \text{Bs}(\mathcal{D}) := \bigcap_{\phi \in \Gamma^\times(\mathcal{X}, \mathcal{D})} \text{Supp}_{\mathbb{R}}(\mathcal{D} + (\phi)), \\ \mathcal{F}(\mathcal{D}) := \sum_{\Gamma} \inf \{ \text{mult}_{\Gamma}(\mathcal{D} + (\phi)) \mid \phi \in \Gamma^\times(\mathcal{X}, \mathcal{D}) \} \cdot \Gamma, \\ \mathcal{P}(\mathcal{D}) := \mathcal{D} - \mathcal{F}(\mathcal{D}), \end{cases}$$

where Γ runs over all prime divisors on \mathcal{X} . Note that the above “inf” can be replaced by “min” because the set $\{ \text{mult}_{\Gamma}(\mathcal{D} + (\phi)) \mid \phi \in \Gamma^\times(\mathcal{X}, \mathcal{D}) \}$ is discrete in \mathbb{R} . The decomposition $\mathcal{D} = \mathcal{P}(\mathcal{D}) + \mathcal{F}(\mathcal{D})$ is called the *sectional decomposition* of \mathcal{D} .

Lemma A.2.1. (1) *The natural inclusion map*

$$H^0(\mathcal{X}, \mathcal{P}(\mathcal{D})) \rightarrow H^0(\mathcal{X}, \mathcal{D})$$

is bijective.

(2) $\text{codim Bs}(\mathcal{P}(\mathcal{D})) \geq 2$.

Proof. By our construction, $\mathcal{D} + (\phi) \geq \mathcal{F}(\mathcal{D})$ for all $\phi \in \Gamma^\times(\mathcal{X}, \mathcal{D})$. Thus (1) follows. Moreover, if $\text{codim Bs}(\mathcal{P}(\mathcal{D})) = 1$, then there is a prime divisor Γ such that

$$\text{mult}_{\Gamma}(\mathcal{P}(\mathcal{D}) + (\phi)) > 0$$

for all $\phi \in \Gamma^\times(\mathcal{X}, \mathcal{D})$, that is, $\text{mult}_{\Gamma}(\mathcal{D} + (\phi)) > \text{mult}_{\Gamma}(\mathcal{F}(\mathcal{D}))$ for all $\phi \in \Gamma^\times(\mathcal{X}, \mathcal{D})$, which is a contradiction. \square

From now on, we assume that $\Gamma_{\mathbb{Q}}^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset$. We set

$$N(\mathcal{D}) := \{ m \in \mathbb{Z}_{\geq 1} \mid \Gamma^\times(\mathcal{X}, m\mathcal{D}) \neq \emptyset \}.$$

Note that $N(\mathcal{D}) \neq \emptyset$. For $m \in N(\mathcal{D})$, we set $\mathcal{F}_m := \mathcal{F}(m\mathcal{D})$ and $\mathcal{P}_m := \mathcal{P}(m\mathcal{D})$.

Lemma A.2.2. (1) $\mathcal{F}_m + \mathcal{F}_{m'} \geq \mathcal{F}_{m+m'}$ for $m, m' \in N(\mathcal{D})$. In particular,

$$\inf_{m \in N(\mathcal{D})} \left\{ \frac{\text{mult}_w(\mathcal{F}_m)}{m} \right\} = \lim_{\substack{m \rightarrow \infty, \\ m \in N(\mathcal{D})}} \frac{\text{mult}_w(\mathcal{F}_m)}{m}$$

for all $w \in \text{DVal}_k(\mathcal{X})$ (cf. [29, Chapter 3, 98]).

$$(2) \mu_{\mathbb{Q}, \Gamma}(\mathcal{D}) = \inf_{m \in N(\mathcal{D})} \left\{ \frac{\text{mult}_{\Gamma}(\mathcal{F}_m)}{m} \right\} \text{ for all prime divisors } \Gamma \text{ on } \mathcal{X}.$$

Proof. (1) is obvious because $\phi \phi' \in \Gamma^\times(\mathcal{X}, (m + m')\mathcal{D})$ for all $\phi \in \Gamma^\times(\mathcal{X}, m\mathcal{D})$ and $\phi' \in \Gamma^\times(\mathcal{X}, m'\mathcal{D})$. For (2), note that

$$\Gamma_{\mathbb{Q}}^\times(\mathcal{X}, \mathcal{D}) = \bigcup_{m \in N(\mathcal{D})} (\Gamma^\times(\mathcal{X}, m\mathcal{D}))^{1/m}.$$

□

A.3. Characterization in terms of μ_w . The following theorem is a characterization of relatively nef Cartier divisors in terms of the asymptotic multiplicity.

Theorem A.3.1. *Let \mathcal{X} be a $(d + 1)$ -dimensional, proper and normal variety over k° and let \mathcal{D} be an \mathbb{R} -Cartier divisor on \mathcal{X} . If $\Gamma_{\mathbb{Q}}^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset$ and $\mu_{\mathbb{Q}, w}(\mathcal{D}) = 0$ for all $w \in \text{DVal}_k(\mathcal{X})$, then \mathcal{D} is relatively nef. In particular, if \mathcal{D} is big, then the following are equivalent:*

- (1) \mathcal{D} is relatively nef with respect to $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$.
- (2) $\mu_{\mathbb{Q}, w}(\mathcal{D}) = 0$ for all $w \in \text{DVal}_k(\mathcal{X})$.
- (3) $\mu_{\mathbb{R}, w}(\mathcal{D}) = 0$ for all $w \in \text{DVal}_k(\mathcal{X})$.

Proof. Let us begin with the following claim:

Claim A.3.1.1. *Let \mathcal{Y} be a normal and proper variety over k° and let $v : \mathcal{Y} \rightarrow \mathcal{X}$ be a dominant morphism over $\text{Spec}(k^\circ)$ such that $\text{Rat}(\mathcal{Y})$ is algebraic over $\text{Rat}(\mathcal{X})$. If*

$$\Gamma_{\mathbb{Q}}^\times(\mathcal{X}, \mathcal{D}) \neq \emptyset \quad \text{and} \quad \mu_{\mathbb{Q}, w}(\mathcal{D}) = 0$$

for all $w \in \text{DVal}_k(\mathcal{X})$, then $\mu_{\mathbb{Q}, w'}(v^(\mathcal{D})) = 0$ for all $w' \in \text{DVal}_k(\mathcal{Y})$.*

Proof. Let w' be a divisorial valuation of $\text{Rat}(\mathcal{Y})$ over k and let w be the restriction of w' to $\text{Rat}(\mathcal{X})$. As $\text{Rat}(\mathcal{Y})$ is algebraic over $\text{Rat}(\mathcal{X})$, we can see that $\mathcal{O}_{w'}/m_{w'}$ is algebraic over \mathcal{O}_w/m_w , so that w is a divisorial valuation of $\text{Rat}(\mathcal{X})$ over k . Then, for an \mathbb{R} -Cartier divisor \mathcal{L} on \mathcal{X} , $\text{mult}_{w'}(v^*(\mathcal{L})) = \text{mult}_w(\mathcal{L})$. Thus,

$$\begin{aligned} \mu_{\mathbb{Q}, w'}(v^*(\mathcal{D})) &= \inf \left\{ \text{mult}_{w'}(v^*(\mathcal{D}) + (\psi)) \mid \psi \in \Gamma_{\mathbb{Q}}^\times(\mathcal{Y}, v^*(\mathcal{D})) \right\} \\ &\leq \inf \left\{ \text{mult}_{w'}(v^*(\mathcal{D} + (\phi))) \mid \phi \in \Gamma_{\mathbb{Q}}^\times(\mathcal{X}, \mathcal{D}) \right\} \\ &= \inf \left\{ \text{mult}_w(\mathcal{D} + (\phi)) \mid \phi \in \Gamma_{\mathbb{Q}}^\times(\mathcal{X}, \mathcal{D}) \right\} = \mu_{\mathbb{Q}, w}(\mathcal{D}), \end{aligned}$$

which prove the claim. □

Claim A.3.1.2. *We may assume that \mathcal{X} is regular and projective over $\text{Spec}(k^\circ)$.*

Proof. We assume that the theorem holds if \mathcal{X} is regular and projective. By de Jong's theorem [13], there is a regular and projective variety \mathcal{Y} over k° together with a dominant morphism $\mu : \mathcal{Y} \rightarrow \mathcal{X}$ over $\text{Spec}(k^\circ)$ such that $\text{Rat}(\mathcal{Y})$ is algebraic over $\text{Rat}(\mathcal{X})$. By the previous claim and our assumption, we can see that $v^*(\mathcal{D})$ is relatively nef, so that \mathcal{D} is also relatively nef. □

Let C be an irreducible and reduced curve on \mathcal{X}_\bullet . Let us see $(\mathcal{D} \cdot C) \geq 0$. Clearly we may assume that \mathcal{D} is effective and $C \subseteq \text{Supp}_\mathbb{R}(\mathcal{D})$. There is a succession of blowing-ups $\rho : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ at closed points such that the strict transform \tilde{C} of C is regular (cf. [17, Theorem 1.101]). If $(\rho^*(\mathcal{D}) \cdot \tilde{C}) \geq 0$, then $(\mathcal{D} \cdot C) = (\rho^*(\mathcal{D}) \cdot \tilde{C}) \geq 0$, so that we may assume that C is regular.

Let $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ be the blowing-up along C and let \mathcal{E} be the exceptional set of π . Let \mathcal{D}' be the strict transform of \mathcal{D} . Then $\pi^*(\mathcal{D}) = \mathcal{D}' + e\mathcal{E}$ for some $e \in \mathbb{Z}_{>0}$. Let H be a very ample divisor on \mathcal{E} . Choosing general members H_1, \dots, H_{d-1} of $|H|$, we set $C' = H_1 \cap \dots \cap H_{d-1}$ and $\pi_*(C') = aC$ for some $a \in \mathbb{Z}_{>0}$. As H_1, \dots, H_{d-1} are general, $C' \not\subseteq \text{Supp}_\mathbb{R}(\mathcal{D}') \cap \mathcal{E}$. If $(\mathcal{E} \cdot C') = (\mathcal{O}_\mathcal{X}(\mathcal{E})|_\mathcal{E} \cdot H^{d-1}) \geq 0$, then

$$a(\mathcal{D} \cdot C) = (\pi^*(\mathcal{D}) \cdot C') = (\mathcal{D}' \cdot C') + e(\mathcal{E} \cdot C') \geq 0.$$

Thus we may assume $(\mathcal{O}_\mathcal{X}(\mathcal{E})|_\mathcal{E} \cdot H^{d-1}) < 0$.

Let $m\pi^*(\mathcal{D}) = \mathcal{P}_m + \mathcal{F}_m$ be the sectional decomposition of $m\pi^*(\mathcal{D})$. By virtue of (1) in Lemma A.2.2, there are finitely many prime divisors $\Gamma_1, \dots, \Gamma_r$ of \mathcal{Y} such that

$$\mathcal{F}_m = a_{m,1}\Gamma_1 + \dots + a_{m,r}\Gamma_r$$

for some $a_{m,1}, \dots, a_{m,r} \in \mathbb{R}_{\geq 0}$. First we assume that $\Gamma_i \neq \mathcal{E}$ for all $i = 1, \dots, r$. Choosing general members $H_1, \dots, H_{d-2} \in |H|$ if necessarily, we have

$$C' \not\subseteq (\text{Bs}(\mathcal{P}_m) \cup \Gamma_1 \cup \dots \cup \Gamma_r) \cap \mathcal{E},$$

and hence

$$am(\mathcal{D} \cdot C) = (\pi^*(m\mathcal{D}) \cdot C') = (\mathcal{P}_m \cdot C') + \sum_{i=1}^r a_{m,i}(\Gamma_i \cdot C') \geq 0.$$

Therefore, we may assume that $\Gamma_1 = \mathcal{E}$. By (2) in Lemma A.2.2 and our assumption, $\lim_{m \rightarrow \infty} a_{m,1}/m = 0$. For any $\epsilon > 0$, we choose m such that $0 \leq a_{m,1}/m \leq \epsilon$. As before, choosing general members $H_1, \dots, H_{d-2} \in |H|$ if necessarily,

$$C' \not\subseteq (\text{Bs}(\mathcal{P}_m) \cup \Gamma_2 \cup \dots \cup \Gamma_r) \cap \mathcal{E}$$

holds, so that $(\mathcal{P}_m \cdot C') \geq 0$ and $(\Gamma_i \cdot C') \geq 0$ for $i = 2, \dots, r$. Thus

$$a(\mathcal{D} \cdot C) = (\pi^*(\mathcal{D}) \cdot C') \geq (a_{m,1}/m)(\mathcal{E} \cdot C') \geq \epsilon(\mathcal{O}_\mathcal{X}(\mathcal{E})|_\mathcal{E} \cdot H^{d-2}).$$

Therefore $(\mathcal{D} \cdot C) \geq 0$ because ϵ is an arbitrary small number.

Finally, the last assertion of the theorem follows from the first assertion, (7) and (8) in Proposition A.1.1. \square

As a corollary, we have the following characterization of relatively nef \mathbb{R} -Cartier divisors. It is proved in [6, Theorem 5.11 and Lemma 5.12] in the case where the characteristic of k°/k° is zero. In general, it seems to be proved by Thuillier. Note that our proof is based on de Jong's alteration.

Corollary A.3.2. *Let X be a proper and normal variety over k and let L be an \mathbb{R} -Cartier divisor on X . Let \mathcal{X} be a normal model of X over $\text{Spec}(k^\circ)$ and let \mathcal{L} be an \mathbb{R} -Cartier on \mathcal{X} with $\mathcal{L} \cap X = L$. We assume there is a sequence $\{(\mathcal{X}_n, \mathcal{L}_n)\}_{n=1}^\infty$ with the following properties:*

- (1) \mathcal{X}_n is a normal model of X over $\text{Spec}(k^\circ)$.
- (2) \mathcal{L}_n is a relatively nef \mathbb{R} -Cartier divisor on \mathcal{X}_n such that $\mathcal{L}_n \cap X = L$.
- (3) $\lim_{n \rightarrow \infty} \text{mult}_w(\mathcal{L}_n) = \text{mult}_w(\mathcal{L})$ for all $w \in \text{DVal}_k(\mathcal{X})$.

Then \mathcal{L} is relatively nef.

Proof. If v is trivial, then the assertion is obvious, so that we assume that v is non-trivial. Clearly we may assume that there is a birational morphism $v_n : \mathcal{X}_n \rightarrow \mathcal{X}$ over k° . By using Chow's lemma, we have a birational morphism $\mu : \mathcal{X}' \rightarrow \mathcal{X}$ over k° such that \mathcal{X}' is projective over $\text{Spec}(k^\circ)$. Let X' be the generic fiber of $\mathcal{X}' \rightarrow \text{Spec}(k^\circ)$. Let \mathcal{X}'_n be the normalization of the main component of $\mathcal{X}_n \times_{\mathcal{X}} \mathcal{X}'$, and let $\mu_n : \mathcal{X}'_n \rightarrow \mathcal{X}_n$ be the induced morphism. We set

$$\mathcal{L}' := \mu^*(\mathcal{L}), \quad L' := \mathcal{L}' \cap X' \quad \text{and} \quad \mathcal{L}'_n := \mu_n^*(\mathcal{L}_n).$$

Then \mathcal{X}'_n is a model of X' over $\text{Spec}(k^\circ)$ and \mathcal{L}'_n is a relatively nef \mathbb{R} -Cartier divisor on \mathcal{X}'_n such that $\mathcal{L}'_n \cap X' = L'$. Moreover,

$$\text{mult}_w(\mathcal{L}') = \text{mult}_w(\mathcal{L}) \quad \text{and} \quad \text{mult}_w(\mathcal{L}'_n) = \text{mult}_w(\mathcal{L}_n)$$

for all $w \in \text{DVal}_k(\mathcal{X})$. Therefore, we may assume that \mathcal{X} is projective.

Let \mathcal{A} be a relatively nef and big Cartier divisor on \mathcal{X} . As L is nef on X , $\mathcal{L} + \epsilon \mathcal{A}$ is nef and big on X for $\epsilon > 0$, so that, by virtue of Theorem A.3.1, it is sufficient to see that

$$\mu_{\mathbb{R},w}(\mathcal{L} + \epsilon \mathcal{A}) = 0$$

for all $w \in \text{DVal}_k(\mathcal{X})$ and $\epsilon > 0$. Replacing \mathcal{X} by a suitable birational model, we may assume that there is a vertical prime divisor Γ on \mathcal{X} such that $w = a \text{ord}_\Gamma$ for some positive number a . Let $\mathcal{X}_\circ = a_1 \Gamma_1 + \cdots + a_r \Gamma_r$ be the irreducible decomposition of the central fiber \mathcal{X}_\circ of $\mathcal{X} \rightarrow \text{Spec}(k^\circ)$ as a Weil divisor. Renumbering $\Gamma_1, \dots, \Gamma_r$, we may set $\Gamma = \Gamma_1$. Let w_{Γ_i} be the additive valuation over k arising from Γ_i . Note that $w = w_{\Gamma_1}$. For a positive number δ , there is N such that

$$|\text{mult}_{w_{\Gamma_i}}(\mathcal{L}) - \text{mult}_{w_{\Gamma_i}}(\mathcal{L}_n)| \leq a_i \delta$$

for all $n \geq N$ and $i = 1, \dots, r$. Then $(v_n)_*(\mathcal{L}_n) - \delta \mathcal{X}_\circ \leq \mathcal{L}$ for $n \geq N$. Therefore, as $\mathcal{L}_n - \delta(\mathcal{X}_n)_\circ + \epsilon v_n^*(\mathcal{A})$ is relatively nef and big, by (2), (6.2) and (7) in Proposition A.1.1,

$$\begin{aligned} 0 &\leq \mu_{\mathbb{R},w}(\mathcal{L} + \epsilon \mathcal{A}) \\ &\leq \mu_{\mathbb{R},w}((v_n)_*(\mathcal{L}_n) - \delta \mathcal{X}_\circ + \epsilon \mathcal{A}) \\ &\quad + \text{mult}_w(\mathcal{L} - (v_n)_*(\mathcal{L}_n) + \delta \mathcal{X}_\circ) \quad (\because (2)) \\ &= \mu_{\mathbb{R},w}((v_n)_*(\mathcal{L}_n - \delta(\mathcal{X}_n)_\circ + \epsilon v_n^*(\mathcal{A}))) \\ &\quad + \text{mult}_w(\mathcal{L} - (v_n)_*(\mathcal{L}_n) + \delta \mathcal{X}_\circ) \\ &\leq \mu_{\mathbb{R},w}(\mathcal{L}_n - \delta(\mathcal{X}_n)_\circ + \epsilon v_n^*(\mathcal{A})) \\ &\quad + \text{mult}_w(\mathcal{L} - (v_n)_*(\mathcal{L}_n) + \delta \mathcal{X}_\circ) \quad (\because (6.2)) \\ &= \text{mult}_w(\mathcal{L} - (v_n)_*(\mathcal{L}_n) + \delta \mathcal{X}_\circ) \quad (\because (7)) \\ &\leq |\text{mult}_w(\mathcal{L}) - \text{mult}_w(\mathcal{L}_n)| + \delta \text{mult}_w(\mathcal{X}_\circ) \leq 2a_1 \delta \end{aligned}$$

for $n \geq N$. Thus $\mu_{\mathbb{R},w}(\mathcal{L} + \epsilon \mathcal{A}) = 0$ because δ is an arbitrary positive number. \square

REFERENCES

- [1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley (1969).
- [2] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical surveys and monographs, No. **33**, AMS, (1990).
- [3] J. I. Burgos Gil, P. Philippon and M. Sombra, Arithmetic geometry of toric varieties. Metrics, measures and heights, preprint (arXiv:1105.5584v2 [math.AG]).
- [4] J. I. Burgos Gil, A. Moriawaki, P. Philippon and M. Sombra, Arithmetic positivity on toric varieties, preprint (arXiv:1210.7692v1 [math.AG]).
- [5] S. Boucksom and H. Chen, Okounkov bodies of filtered linear series, *Compositio Math.* **147** (2011), 1205–1229.
- [6] S. Boucksom, C. Favre and M. Jonsson, Singular semipositive metrics in non-archimedean geometry, preprint (arXiv:1201.018v1 [math.AG]).
- [7] H. Chen, Positive degree and arithmetic bigness, preprint (arXiv:0803.2583 [math.AG]).
- [8] H. Chen, Arithmetic Fujita approximation, *Ann. Sci. École Norm. Sup.*, **43** (2010), 555–578.
- [9] G. Faltings, Calculus on arithmetic surfaces, *Ann. of Math.* **119** (1984), 387–424.
- [10] H. Gillet and C. Soulé, An arithmetic Riemann-Roch theorem, *Invent. Math.* **110** (1992), 473–543.
- [11] A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algébrique (EGA)*, Publ. Math. IHES, **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32**, 1960–1967.
- [12] W. Gubler, Local heights of subvarieties over non-archimedean fields, *J. Reine Angew. Math.*, **498** (1998), 61–113.
- [13] A. J. de Jong, Smoothness, semi-stability and alterations, *Publications Mathématiques I.H.E.S.*, **83** (1996), 51–93.
- [14] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. **52**. Springer, New York, (1977).
- [15] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Graduate Texts in Mathematics, No. **25**. Springer, New York, (1975).
- [16] H. Ikoma, Boundedness of the successive minima on arithmetic varieties, *J. Algebraic Geometry*, **22** (2013), 249–302.
- [17] J. Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, Princeton, (2007).
- [18] J. Lipman, Desingularization of two-dimensional schemes, *Ann. of Math.*, **107** (1978), 151–207.
- [19] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics **6**, Oxford University Press (2002).
- [20] A. Moriawaki, The continuity of Deligne’s pairing. *Internat. Math. Res. Notices* 1999, no. 19, 1057–1066.
- [21] A. Moriawaki, Arithmetic height functions over finitely generated fields, *Invent. Math.* **140** (2000), 101–142.
- [22] A. Moriawaki, Continuity of volumes on arithmetic varieties, *J. Algebraic Geom.* **18** (2009), 407–457.
- [23] A. Moriawaki, Continuous extension of arithmetic volumes, *International Mathematics Research Notices*, (2009), 3598–3638.
- [24] A. Moriawaki, Estimation of arithmetic linear series, *Kyoto J. of Math.* **50** (Memorial issue of Professor Nagata) (2010), 685–725.
- [25] A. Moriawaki, Arithmetic linear series with base conditions, *Math. Z.* (DOI) 10.1007/s00209-012-0991-2.
- [26] A. Moriawaki, Zariski decompositions on arithmetic surfaces, *Publ. Res. Inst. Math. Sci.* **48** (2012), 799–898.
- [27] A. Moriawaki, Toward Dirichlet’s unit theorem on arithmetic varieties, to appear in *Kyoto J. of Math.* (Memorial issue of Professor Maruyama), see also (arXiv:1010.1599v4 [math.AG]).
- [28] A. Moriawaki, Numerical characterization of nef arithmetic divisors on arithmetic surfaces, to appear in *Annales de la faculté des sciences de Toulouse*, see also (arXiv:1201.6124 [math.AG]).
- [29] G. Pólya and G. Szegő, *Problems and Theorems in Analysis*, Springer, New York, (1970).
- [30] X. Yuan, Big line bundles over arithmetic varieties, *Invent. math.* **173** (2008), 603–649.
- [31] X. Yuan, On volumes of arithmetic line bundles, *Compositio Mathematicae* **145** (2009), 1447–1464.
- [32] X. Yuan and S. Zhang, Calabi-Yau theorem and algebraic dynamics, preprint.
- [33] S. Zhang, Positive line bundles on arithmetic varieties, *J. of AMS*, **8** (1995), 187–221.

- [34] O. Zariski and P. Samuel, *Commutative algebra II*, Graduate Texts in Mathematics, No. **29**. Springer, New York, (1960).

INDEX

- Bs(\mathcal{D}), 79
- $C^0(M)$, 6
- $C_{F_\infty}^0(X(\mathbb{C}))$, 4, 33
- $C_\eta^0(X^{\text{an}})$, 17
- \mathcal{D}^a , 35
- $\mathcal{D}_{(P)}$, 34
- $D_1 \leq D_2$, 22
- D_W , 9
- $\overline{\mathcal{D}}^a$, 34
- \overline{D}^τ , 34
- $\overline{D}_1 \leq \overline{D}_2$, 34
- DVal $_k(\mathcal{X})$, 76
- Div(X), 7, 8
- Div(X) $_{\mathbb{K}}$, 8
- Div(X) $_{\mathbb{R}}$, 7
- Div(\mathcal{X}), 75
- Div(\mathcal{X}) $_{\mathbb{R}}$, 75
- Div($\mathcal{X}; x$), 75
- Div($\mathcal{X}; x$) $_{\mathbb{R}}$, 75
- Div $_{C^0}^a(X)$ $_{\mathbb{R}}$, 21
- $\widehat{\text{Div}}_{\text{int}}^a(X)$ $_{\mathbb{R}}$, 39
- $\widehat{\text{Div}}_{C^0}^a(X)$ $_{\mathbb{R}}$, 4, 34
- $\mathcal{F}(\mathcal{D})$, 79
- F_∞ , 4, 33
- $G(\overline{D}; V_\bullet)$, 72
- $\Gamma^\times(\mathcal{X}, \mathcal{D})$, 76
- $\Gamma_{\mathbb{K}}^\times(\mathcal{X}, \mathcal{D})$, 76
- $\widehat{\Gamma}_{\mathbb{K}}^\times(X, \overline{D})$, 69
- $H^0(X, D)$, 4, 17
- $H^0(X, D)$, 36
- $H^0(\mathcal{X}, \mathcal{D})$, 2
- $\dot{H}^0(X, \overline{D})$, 36
- $\dot{H}^0(X, \overline{D})$, 5, 36
- K_P , 4, 33
- $\mathbb{M}(X; \Psi)$, 32
- $\mathbb{M}(X^{\text{an}}; \Psi)$, 30
- $N(\mathcal{D})$, 79
- $\mathcal{P}(\mathcal{D})$, 79
- Rat(X), 8
- Rat(X) $_{\mathbb{K}}^\times$, 8, 69
- Rat(\mathcal{X}), 75
- Rat(\mathcal{X}) $_{\mathbb{K}}^\times$, 75
- $\Sigma(\overline{D}; Q)$, 56, 62
- $\Sigma_{\mathcal{X}}(\mathcal{D}; Q)$, 58
- Spec $_k^{\text{an}}(A)$, 10
- Supp $_{\mathbb{K}}(D)$, 8
- Supp $_W(D)$, 9
- $\Theta(\overline{D}; V_\bullet)$, 72
- $\Upsilon(\overline{\mathcal{D}})$, 3
- $\Upsilon(\overline{D})$, 6, 67
- $\Upsilon(\overline{D}; R)$, 63
- $\Upsilon_{\text{rel}}(\overline{D})$, 63
- WDiv(\mathcal{X}), 75
- WDiv(\mathcal{X}) $_{\mathbb{R}}$, 75
- $X(\mathbb{C})$, 4, 33
- $\mathcal{X}_{(P)}$, 34
- \mathcal{X}_o , 15
- X^{an} , 3, 10
- X_P , 4
- $\hat{\chi}(M, \|\cdot\|)$, 6
- $\hat{\chi}(X, \overline{D})$, 37
- $\widehat{\deg}(\overline{\mathcal{D}}_1 \cdots \overline{\mathcal{D}}_{d+1})$, 2
- $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d+1})$, 5, 40
- $\widehat{\deg}(\overline{D}|_x)$, 35
- $\widehat{\deg}_P(\overline{D}|_x)$, 35
- $\widehat{\deg}_v(\mathcal{L}_1 \cdots \mathcal{L}_d; \phi)$, 22
- $\widehat{\deg}_v(\overline{D}|_x)$, 22
- $\widehat{\deg}_v(\overline{L}_1 \cdots \overline{L}_{d+1})$, 27
- $\widehat{\deg}_v(\overline{L}_1 \cdots \overline{L}_d; \phi)$, 25
- $\widehat{\deg}_\infty(\overline{D}|_x)$, 35
- f^{an} , 10
- $g(\mathcal{X}, \mathcal{D})$, 3, 18
- $\hat{h}^0(M, \|\cdot\|)$, 6
- $\hat{h}^0(X, \overline{D})$, 36
- k° , 6
- $k^{\circ\circ}$, 6
- k_ν , 15
- $\log|\mathcal{J}|$, 30
- $\max\{D_1, \dots, D_r\}$, 7
- $\min\{D_1, \dots, D_r\}$, 7
- $\mu_{\mathbb{K}, w}(\mathcal{D})$, 76
- $\mu_{\mathbb{K}, \xi}(\overline{D})$, 69
- mult $_w$, 76
- mult $_{z_P}^\sim$, 71
- $\|\phi\|_{g_\infty}$, 2
- $\|\phi\|_{g_\nu}$, 4
- $\|\phi\|_g$, 17
- ord $_A$, 7
- ord $_\Gamma$, 7
- ord $_\gamma$, 7
- ord $_{z_P}^\sim(f)$, 71
- $(\varphi)_{\mathbb{K}}$, 69
- $(\widehat{\varphi})$, 34
- $(\widehat{\varphi})_{\mathbb{K}}$, 69
- $r_{\mathcal{X}}$, 10
- $\widehat{\text{vol}}(\overline{\mathcal{D}})$, 2
- $\widehat{\text{vol}}(\overline{D})$, 5, 37
- $\widehat{\text{vol}}(\overline{D}; V_\bullet)$, 72
- $\widehat{\text{vol}}_\chi(\overline{D})$, 37
- $\widehat{\text{vol}}_\chi(\overline{D}; V_\bullet)$, 72
- ν_P , 4, 33
- adelic arithmetic principal divisor, 34

- adelic arithmetic \mathbb{R} -Cartier divisor of C^0 -type, 4, 33
- adelic \mathbb{R} -Cartier divisor of C^0 -type, 21
- arithmetic \mathbb{R} -Cartier divisor of C^0 -type, 1
- arithmetic variety, 6
- associated adelic arithmetic \mathbb{R} -Cartier divisor, 34
- associated adelic \mathbb{R} -Cartier divisor, 22
- associated global adelic \mathbb{R} -Cartier divisor, 35
- associated \mathbb{K} -Weil divisor, 8
- associated prime of the multiplicative semi-norm, 10
- associated valuation, 10
- big adelic arithmetic \mathbb{R} -Cartier divisor, 37
- big \mathbb{R} -Cartier divisor, 76
- birational system of models, 30
- center of valuation, 76
- $\hat{\chi}$ -volume of adelic arithmetic \mathbb{R} -Cartier divisor, 37
- defining model, 34
- divisorial valuation, 76
- fractional ideal sheaf, 29
- global adelic \mathbb{R} -Cartier divisor of C^0 -type, 34
- global degree, 36
- global model function, 32
- Green function induced by model, 3
- Green function of C^0 -type, 3, 17
- integrable adelic arithmetic \mathbb{R} -Cartier divisor, 5, 39
- integrable adelic \mathbb{R} -Cartier divisor, 21
- integrable arithmetic \mathbb{R} -Cartier divisor, 2
- \mathbb{K} -asymptotic multiplicity, 69, 76
- \mathbb{K} -Cartier divisor, 8
- \mathbb{K} -principal divisor, 8
- \mathbb{K} -rational function, 8
- \mathbb{K} -support, 8
- local degree, 22
- model, 6
- model function, 30
- model of (X, D) , 3, 18
- model of D , 18
- multiplicative semi-norm, 9
- nef adelic arithmetic \mathbb{R} -Cartier divisor, 38
- normal model, 6
- of $(C^0 \cap \text{PSH})$ -type, 3, 20
- pseudo-effective arithmetic \mathbb{R} -Cartier divisor, 37
- \mathbb{R} -Cartier divisor, 7
- reduction map, 10
- regular model, 6
- relatively nef, 7
- relatively nef adelic arithmetic \mathbb{R} -Cartier divisor, 37
- relatively nef adelic \mathbb{R} -Cartier divisor, 21
- relatively nef global adelic \mathbb{R} -Cartier divisor, 39
- residue field, 10
- sectional decomposition, 79
- support as a Weil-divisor, 9
- truncation of the adelic arithmetic \mathbb{R} -Cartier divisor, 34
- variety over a noetherian integral scheme, 6
- vertical curve, 7
- vertical fractional ideal sheaf, 29
- volume of adelic arithmetic \mathbb{R} -Cartier divisor, 37

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN
E-mail address: moriwaki@math.kyoto-u.ac.jp